

First Order Logic Summary

Overview:

First order logic allows us to deal with the internal structure of statements. Previously, we were only concerned with statements joined by truth functional connectives (\wedge, \vee , etc.) where we treated those statements as indivisible building blocks of an argument. Now, in addition to all the machinery we developed for zeroth order logic, we will look inside statements and identify subjects and predicates. A predicate says something about or makes a claim about the subject. Subjects and what is predicated about them can be very general. But one thing we won't do is allow a statement to predicate something about itself. For example, "*This statement is false*" violates the rules of first order logic by setting up self-reference.

Definitions:

Subject - the part of a statement that names a thing or things about which some claim is to be made or property asserted. In the statement "Socrates is mortal", Socrates is the subject.

Predicate - the part of a statement which makes a claim about the subject or asserts that the subject possesses some property. In the statement "Socrates is mortal", "is mortal" is the predicate.

Propositional function - a statement that contains variables instead of constants. It has no truth value until constants are substituted for the variables.

Singular proposition - a statement where the subject is a constant. The statement "Socrates is mortal" is a singular proposition about the specific man Socrates.

General proposition - a statement where the subject is a quantified variable. The statement "All men are mortal" assigns mortality to the variable which represents an arbitrary man.

Substitution instance - replacing a variable in a propositional function with a constant. This is analogous to evaluating a math function at a particular value.

Instantiation - passing from a quantified propositional function to a proposition by making a substitution instance.

Generalization - passing from a proposition to a quantified propositional function

Universal instantiation (UI) - passing from the statement that a variable object has a given property to any substitution instance of that variable. From "All cows are brown", we may substitute the cow "Bossy" to conclude "Bossy is brown".

Existential instantiation (EI) - passing from the statement that there are some objects that have a given property to a particular substitution instance (a true one) of that variable. From "Some cows are brown", we may not substitute the specific cow "Bossy" to conclude "Bossy is brown". We may only state "A brown cow exists". We cannot control which cow is brown with existential quantification.

Universal generalization (UG) - passing from a particular object having a given property to the statement that all objects have the given property. This relies on the arbitrariness of

the particular object.

Existential generalization (EG) - passing from a particular object having a given property to the statement that some objects have the given property.

Notation:

(x) or $(\forall x)$ means "for all x " - this is called the *universal* quantifier - negation is written simply $\neg(x)$

$(\exists x)$ means "for some x " - this is called the *existential* quantifier - negation is $\neg(\exists x)$

Constants will be represented by the letters a, \dots, w

Variables will be represented by $x, y,$ and z

Properties will be represented by capital letters A, \dots, Z

The fact that a variable x or constant a has property P will be written Px or $Pa,$ respectively

Note, for example, that $(x)Ax \supset Bx$ does not mean the same as $(x)(Ax \supset Bx),$ so use parentheses accordingly

To negate $(x)Px$ use $\neg(x)Px,$ and not $(x)\neg Px,$ since $(x)\neg Px$ means absolutely no x have property $P.$ All it takes is one x without property P to negate $(x)Px.$

To negate $(\exists x)Px$ use $\neg(\exists x)Px,$ and not $(\exists x)\neg Px,$ since $(\exists x)\neg Px$ just means at least one x does not have property $P.$ It could be the case that some x have it and some don't. The negation has to rule out any x having property $P.$

The secondary variable y will often be used as the "arbitrary" object that leads to a universal generalization in terms of $x...$ consider it a parameter, or unspecified constant. There is no magic in the particular selection of " y ".

Basic Implications:

$(x)Px$ always implies $(\exists x)Px$ and likewise $(x)\neg Px$ always implies $(\exists x)\neg Px$

$\neg(x)Px$ always implies $(\exists x)\neg Px$ and $\neg(\exists x)\neg Px$ always implies $(\exists x)Px$ as contradictories, so by contraposition, $\neg(\exists x)Px$ implies $\neg[\neg(x)\neg Px] \equiv (x)\neg Px$ (see remarks above on negation of quantifiers)

Scope:

Universal instantiation and existential generalization present no difficulties regarding scope.

Existential instantiation can go wrong the following way. Suppose the constant a appears in an argument prior to an *EI* and it is shown to have property $P.$ If we now instantiate with respect to the property Q and use a as the individual constant in the instantiation, we have gratuitously given the object $a,$ known to have property $P,$ also the property $Q.$ There is no reason to think that the object acquiring property Q by instantiation is precisely the same object that earlier was shown to have property $P.$ To avoid this, *use a constant for each existential instantiation that has not appeared before in the argument.*

Universal generalization can go wrong two ways. First, the idea behind *UG* is that once an arbitrary object is shown to have a property, the arbitrariness allows us to conclude that all objects have the given property. We can mistakenly use an object that is not arbitrary. "Lead is heavy" therefore "Everything is heavy" is an invalid argument because lead is a

specific and not arbitrary object.. Second, in a conditional proof where we make an assumption which has scope extending to the discharge of the assumption (where the conclusion is a conditional statement with the assumption as antecedent), there is a danger of establishing within the scope of an assumption something about an "arbitrary" object, then generalizing that outside the scope of the assumption. For example, if within a conditional proof we assumed "A bunny is cute", we could then incorrectly generalize to "Everything is cute", then discharge the assumption by concluding "If a bunny is cute, everything is cute". Here the bunny is an arbitrary object, but the cuteness was assumed, not proved outside the scope of that assumption. So the caveat to avoid problems with *UG* is to make sure (i) that the object being generalized is truly arbitrary, and (ii) the assertion that the arbitrary object has the property being generalized does not appear in the scope of any assumption involving that object in any way. Some examples may clarify.

Examples:

Universal Instantiation:

Given $(x)\Phi(x)$, conclude Φc for any constant c

You are saying that if every substitution instance of the variable x has property Φ , then the substitution of the particular constant c requires c to have property Φ .

All dogs have fleas. Fido is a dog. Therefore, Fido has fleas.

1) $(x)(Dx \supset Fx)$

2) $Df \quad / \quad \therefore Ff$

3) $Df \supset Ff \quad 1, UI$

4) $Ff \quad 2, 3, MP$

Existential Generalization:

Given Φc for any constant c , conclude $(\exists x)\Phi x$

You are saying that if the constant c has property Φ , then at least one value of the variable x (namely c) has property Φ

If Fido has fleas, he will scratch himself. Fido is a dog and has fleas. Therefore, some dog will scratch himself.

1) $Ff \supset Sf$

2) $Df \wedge Ff \quad / \quad \therefore (\exists x)(Dx \wedge Sx)$

3) $Ff \wedge Df \quad 2, Com$

4) $Ff \quad 3, Sim$

5) $Sf \quad 1, 4, MP$

6) $Df \quad 2, Sim$

7) $Df \wedge Sf \quad 5, 6, Con$

8) $(\exists x)(Dx \wedge Sx) \quad 7, EG$

Existential Instantiation:

Given $(\exists x)\Phi x$, conclude Φc provided c has not appeared prior in the argument

You are saying that since at least one value of the variable x has property Φ , then we may choose it to be c provided c has not yet appeared in the argument. If c has already appeared, by re-using it for the instantiation you are gratuitously giving c the property Φ that

it may not really have.

We also have to pay attention to the little matter of using y as an arbitrary constant (parameter), since the convention is that universal generalization, discussed below, uses the fact that y has property Φ to conclude that all x have property Φ . It is important that y be completely arbitrary for this to work. If we allow y to be instantiated as a constant, it really isn't arbitrary. Here is an argument that illustrates this:

Some dogs have fleas. Therefore, all dogs have fleas.

- 1) $(\exists x)(Dx \wedge Fx) \quad / \therefore (x)(Dx \supset Fx)$
- 2) $Dy \wedge Fy \quad 1, EI$
- 3) $Fy \wedge Dy \quad 2, Com$
- 4) $Fy \quad 3, Sim$
- 5) $Fy \vee \neg Dy \quad 4, Add$
- 6) $\neg Dy \vee Fy \quad 5, Com$
- 7) $Dy \supset Fy \quad 6, MI$
- 8) $(x)(Dx \supset Fx) \quad 7, UG$

The error lies in instantiating y , which is now not arbitrary, since it was one of the dogs with fleas...some dogs still don't have fleas. Then in line 8, we ignore the non-arbitrariness of y and generalize. So the rule is *you can't universally generalize from a constant that was gotten by an existential instantiation*.

(Bogus) Some animals have wings. Fido is an animal. Therefore, Fido has wings.

- 1) $(\exists x)(Ax \wedge Wx)$
- 2) $Af \quad / \therefore Wf$
- 3) $Af \wedge Wf \quad 1, EI$
- 4) $Wf \wedge Af \quad 3, Com$
- 5) $Wf \quad 4, Sim$

The invalid step is 3), since the constant f being instantiated has already appeared in the argument at line 2. You are pasting wings on Fido by using f instead of a new constant (who would have wings but not necessarily be Fido).

(Correct, but misleading) Some animals have wings. Therefore, Fido has wings.

- 1) $(\exists x)(Ax \wedge Wx) \quad / \therefore Wf$
- 2) $Af \wedge Wf \quad 1, EI$
- 3) $Wf \wedge Af \quad 2, Com$
- 4) $Wf \quad 3, Sim$

This is OK...you are referring to the animal with wings given by instantiating the premiss as Fido. Fido may be a bird.

(Correct, not misleading) Some of the dogs we have are terriers. We also have Fido, who is a hound. Therefore, we have both a terrier and a hound.

- 1) $(\exists x)Tx$
- 2) $Hf \quad / \therefore Ta \wedge Hf$
- 3) $Ta \quad 1, EI$
- 4) $Ta \wedge Hf \quad 2, 3, Con$

The dog a here is a different dog than Fido f

Universal Generalization:

Given Φ_y for any arbitrary constant (parameter) y , conclude $(x)\Phi_x$ provided y is *truly* arbitrary and Φ_y never appears in the scope of any assumption (as in a conditional proof) involving y .

You are saying that if an arbitrary constant y can be shown to possess property Φ , and this fact Φ_y has not been created by assuming it as part of a conditional proof, then it is valid to conclude that all constants have property Φ , which is the same as saying all substitution instances of the variable x have property Φ .

(Bogus, because the parameter is not arbitrary) Fido is a big dog. Therefore, everything is a big dog.

- 1) $Df \wedge Bf \quad / \therefore (x)(Dx \wedge Bx)$
- 2) $(x)(Dx \wedge Bx) \quad 1, UG$

If somehow the statement $Dy \wedge By$ had appeared, where y represented an arbitrary constant and not specifically Fido, the conclusion would have been correct, if preposterous. What sort of argument would that entail? Every object is a big dog. Therefore, everything is a big dog. Well, yes, I suppose.

(Bogus, because we have gratuitously assumed a property into existence) Not all dogs have fleas. Therefore, no dog whatsoever has fleas.

- 1) $\neg(x)(Dx \supset Fx) \quad / \therefore (x)\neg(Dx \supset Fx)$
- 2) $Dy \supset Fy \quad \text{assumption}$
- 3) $(x)(Dx \supset Fx) \quad 2, UG$

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- 4) $(Dy \supset Fy) \supset (x)(Dx \supset Fx) \quad 2, 3, CP$
 - 5) $\neg(Dy \supset Fy) \quad 1, 4, MT$
 - 6) $(x)\neg(Dx \supset Fx) \quad 5, UG$

Here the property Φ_y being generalized is $Dy \supset Fy$. Although y is certainly arbitrary, appearing for the first time in line 2, Φ_y appears inside the scope of a conditional proof assumption...in fact, *it is the assumption*. The mistake is using universal generalization to extend this assumption Φ_y to all x in line 3. We have no business assuming something and then pretending it applies across the board.

Review the argument above in the existential instantiation section which presents another type of error...you cannot universally generalize a constant that has been previously existentially instantiated. This leads to arguments of the form "Some dogs bite, therefore all dogs bite".

(Correct) No dogs can read. All beagles are dogs. Therefore, no beagles can read.

- 1) $(x)(Dx \supset \neg Rx)$
- 2) $(x)(Bx \supset Dx) \quad / \therefore (x)\neg(Bx \supset \neg Rx)$
- 3) $By \supset Dy \quad 2, UI$
- 4) $Dy \supset \neg Ry \quad 1, UI$
- 5) $By \supset \neg Ry \quad 3, 4, HS$
- 6) $(x)\neg(Bx \supset \neg Rx) \quad 5, UG$

In lines 3 and 4 we have used the parameter y . Because the instantiations are universal, it is arbitrary, and the generalization in line 6 is valid. If either of the instantiations were existential, using y and subsequently generalizing it universally would have been a mistake.

Summary of Rules:

- 1) UI - use any symbol
- 2) EG - no restriction...remember you can't export anything from the scope of an assumption in CP , so assuming Φ_c doesn't let you conclude $(\exists x)\Phi_x$
- 3) EI - don't use a symbol that has already appeared and don't use a symbol that you are going to universally generalize later...usually we reserve y for the arbitrary constant in UG , so don't use it for EI
- 4) UG - make sure the symbol you are generalizing is arbitrary (usually y , by convention) and don't generalize any property that appears within the scope of a CP assumption involving the arbitrary symbol.