Vector Geometry

① Create a vector between two points:

Given point $P_1 = (a_1, b_1, c_1)$ and point $P_2 = (a_2, b_2, c_2)$, the vector representing the line segment <u>from</u> P_1 <u>to</u> P_2 is

$$\mathbf{v} = \langle a_2 - a_1,\ b_2 - b_1,\ c_2 - c_1 \rangle$$

② Create a unit vector (length = 1) in the direction of a given vector:

Given $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$, the unit vector parallel to $\mathbf{v}$ is $\dfrac{\mathbf{v}}{|\mathbf{v}|}$, or

more specifically,

$$\hat{e}_v = \frac{1}{\sqrt{v_1^2 + v_2^2 + v_3^2}} \langle v_1, v_2, v_3 \rangle$$

3 To find the dot product (or scalar product) between two vectors $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ and $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$, multiply respective components and add:

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$$

Note $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$

If you know the angle between the vectors (θ), then

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos\theta,$$ where

$$|\mathbf{u}| = (u_1^2 + u_2^2 + u_3^2)^{1/2}$$ is the length of $\mathbf{u}$ (likewise for $\mathbf{v}$).

4 To find the cross product (or vector product) between two vectors $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ and $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$

evaluate the "formal" determinant:

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \begin{array}{l} \\ \leftarrow \text{left factor} \\ \leftarrow \text{right factor} \end{array}$$

The determinant is "formal" because the entries are not all scalars. All we want is the evaluation scheme applied to the entries. The first (leftmost) factor has its components in the middle row and the second at the bottom.

Remember $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$. Order matters. Note also that the cross product produces a <u>vector</u> that happens to be perpendicular to both $\mathbf{u}$ and $\mathbf{v}$.

If you know the angle between the vectors (θ), then

$$\mathbf{u} \times \mathbf{v} = |\mathbf{u}||\mathbf{v}| \sin\theta \, \hat{e}_{\perp},$$

where $\hat{e}_{\perp}$ is a unit vector perpendicular to both $\mathbf{u}$ and $\mathbf{v}$ as determined by the right hand rule.

5 To find a vector perpendicular to a given plane, do one of the following:

(i) If the plane is given by its equation in standard form, $ax + by + cz = d$, a perpendicular vector is $\langle a, b, c \rangle$.

(ii) If the plane is given by two vectors (non-parallel) in the plane, say $\mathbf{u}$ and $\mathbf{v}$, the vector $\mathbb{W} = \mathbb{U} \times \mathbb{V}$ will be perpendicular to the plane. If you want to unitize $\mathbb{W}$, calculate $\hat{e}_w := \dfrac{\mathbb{W}}{|\mathbb{W}|}$.

(iii) If the plane is given by three non-collinear [collinear means] (all on 1 line) points, pick one point as the base and find two vectors, one going from the base to each of the other points (see 1). Now apply 5 ii above.

6 To find the angle between two vectors $\mathbb{U} = \langle U_1, U_2, U_3 \rangle$ and $\mathbb{V} = \langle V_1, V_2, V_3 \rangle$ note that $\mathbb{U} \cdot \mathbb{V} = |\mathbb{U}||\mathbb{V}| \cos\theta$.

Solving for θ:

$$\theta = \arccos\left(\frac{\mathbb{U} \cdot \mathbb{V}}{|\mathbb{U}||\mathbb{V}|}\right)$$

or $$\theta = \arccos\left(\frac{u_1v_1 + u_2v_2 + u_3v_3}{(u_1^2+u_2^2+u_3^2)^{1/2}(v_1^2+v_2^2+v_3^2)^{1/2}}\right)$$

(7) To find the (dihedral) angle between two planes, use (5) to get direction vectors for each plane, then use (6) to find the angle between the direction vectors. If the angle is $0°$, the planes are parallel (possibly co-inciding). Otherwise the planes meet in a line.

(8) To find the area of a triangle in space:

(i) If it is given as three non-collinear points, imitate (5) iii and construct vectors $\mathbf{u}$ and $\mathbf{v}$ as sides of the triangle.

Then the area of the triangle is

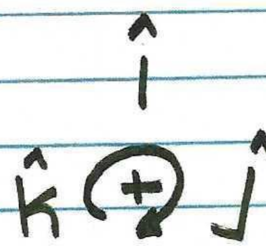
$$\text{Area} = \frac{1}{2}|\mathbf{u} \times \mathbf{v}| \quad \text{(triangle)}$$

For a parallelogram, drop the $\frac{1}{2}$:

$$\text{Area} = |\mathbf{u} \times \mathbf{v}| \quad \text{(parallelogram)}$$

9 To find the volume of a parallelepiped (sheared shoe box) you would be given three non-coplanar vectors or four points in space determining such vectors. If the vectors are A, B, & C, form the scalar triple product $A \cdot (B \times C)$. You may permute these vectors as you wish, and the volume of the figure will be the absolute value of the scalar triple product.

10 A mnemonic device for taking the cross product of the basis vectors in $\mathbb{R}^3$ is as follows:



Proceeding clockwise around the figure gives a positive result and counterclockwise gives a negative. So $\hat{i} \times \hat{j} = +\hat{k}$, but $\hat{j} \times \hat{i} = -\hat{k}$, and so forth.

11 A line in 3-space is determined by a point on it and a vector parallel to it. By analogy with 2-space, the line $y = mx + b$ certainly goes thru the point $(0, b)$ and

has slope m (rise over run). In 3-space, the vector $\mathbf{r}_0$ that connects the origin with the point P_0 is analogous to the y-intercept, and the direction vector $\mathbf{v}$ plays the rôle of m. So we write

$$\mathbf{r}(t) = \mathbf{r}_0 + t\,\mathbf{v}$$

Here $\mathbf{r}(t)$ is the (infinit) line given as a function of the parameter $t \in (-\infty, \infty)$. $\mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle$ connects the origin $(0,0,0)$ to the "starting point" (x_0, y_0, z_0). Finally $\mathbf{v}$ is a vector that captures the direction of the line. Written as components, $\langle x(t), y(t), z(t) \rangle =$

$$\langle x_0, y_0, z_0 \rangle + t \langle v_1, v_2, v_3 \rangle$$

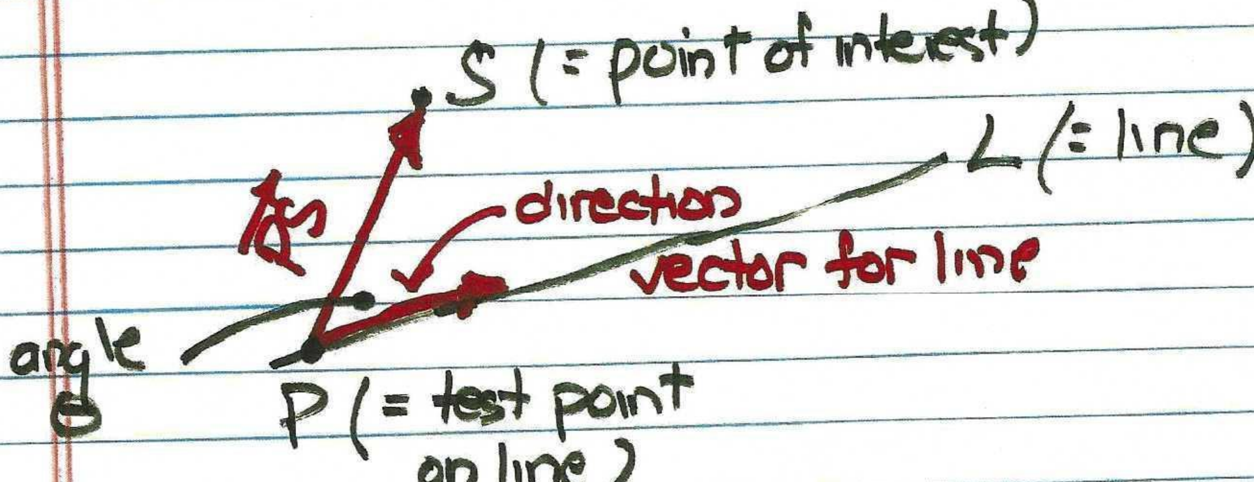
This expression can be decomposed into component expressions:

$$x(t) = x_0 + v_1 t$$

$$y(t) = y_0 + v_2 t$$

$$z(t) = z_0 + v_3 t$$

12 To find the (perpendicular) distance between a line and a point not on it, consider this sketch:



We are given the co-ordinates of S and the equation of the line: $\mathbf{r}(t) = \mathbf{r}_0 + t\,\mathbf{v}$

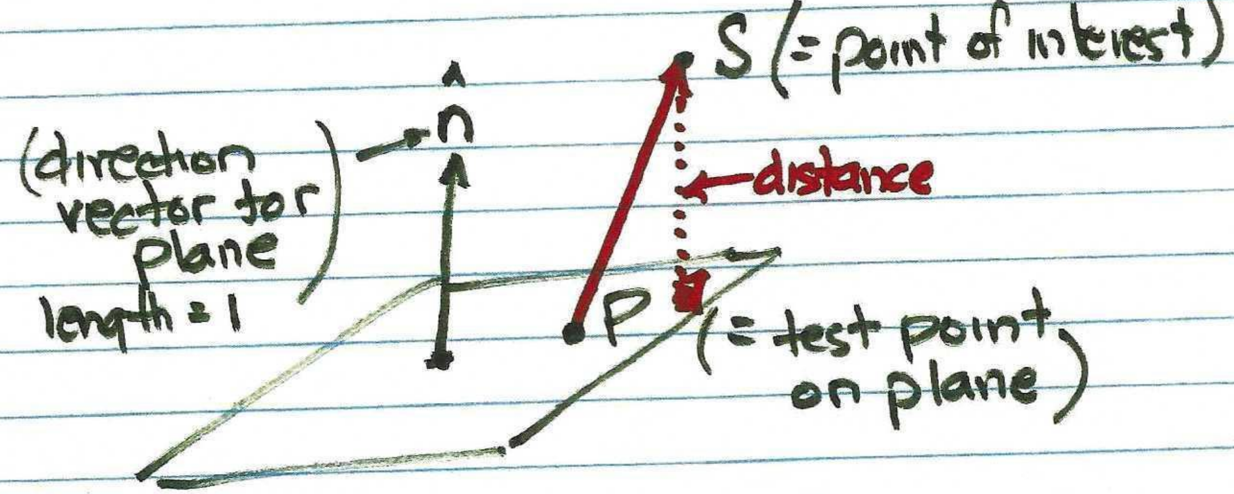
Select some point P on the line by choosing a value for t (usually zero). This gives the co-ordinates (x_0, y_0, z_0) for P.

If the co-ordinates of S are (x_1, y_1, z_1), use 1 to form the vector $\overrightarrow{PS}$ between them. The distance from S to the line is the component of $\overrightarrow{PS}$ perpendicular to the line, namely $|\overrightarrow{PS}|\sin\theta$. We get that by computing $\overrightarrow{PS} \times \mathbf{v}$ and dividing by $|\mathbf{v}|$:

$$\text{distance S to L} = \frac{|\overrightarrow{PS}||\mathbf{v}|\sin\theta}{|\mathbf{v}|}$$

$$= \frac{|\overrightarrow{PS} \times \mathbf{v}|}{|\mathbf{v}|}.$$

(13) To find the (perpendicular) distance between a plane and a point not on it, consider this sketch:



Use (1) to get the vector $\vec{PS}$ from the given co-ordinates of S and a point P chosen on the plane. If the equation of the plane (from (5)) is $ax+by+cz=d$, set $x=y=0$ and solve for z to get z_0. Then P has co-ordinates $(0,0,z_0)$. If the plane is the x-y plane, then $z_0=0$.

The projection of $\overrightarrow{PS}$ parallel to the unit normal direction vector is the distance from S to the plane. The unit normal is $\frac{1}{\sqrt{a^2+b^2+c^2}}\langle a, b, c\rangle = \hat{n}$

So the required distance is:

distance S to plane =

$$\frac{\overrightarrow{PS} \cdot \langle a, b, c\rangle}{\sqrt{a^2+b^2+c^2}}$$

or if you already have $\hat{n}$,

$$\overrightarrow{PS} \cdot \hat{n}$$