TRIG SUBSTITUTIONS, FORWARD AND REVERSE

BACKGROUND:

Trigonometric substitutions often allow us to handle superficially difficult integrals easily. For example, the substitution $x = \tan\theta$ reduces the integral $\int \frac{1}{1+x^2}dx$ to a trivial form. Conversely, some difficult trig integrals, for example $\int \frac{d\theta}{2+\cos\theta}$, become simplified by the *reverse* treatment, where we convert the trig function back to a rational function.

THEORY:

Forward trig substitutions:

Expressions involving $\sqrt{x^2+a^2}$ can be transformed by letting $x = a\tan\theta$, then $dx = a\sec^2\theta d\theta$. This corresponds to a mnemonic right triangle with base angle θ, hypotenuse $\sqrt{x^2+a^2}$, adjacent side a, and opposite side x. For example, $\int \frac{dx}{1+x^2} = \int \frac{\sec^2\theta d\theta}{1+\tan^2\theta} = \int d\theta$.

Changing the right triangle to one with base angle θ, hypotenuse x, adjacent side a, and opposite side $\sqrt{x^2-a^2}$ suggests setting $x = a\sec\theta$ and $dx = a\sec\theta\tan\theta d\theta$. Then, for exmple, $\int \frac{dx}{x\sqrt{x^2-1}} = \int \frac{\sec\theta\tan\theta d\theta}{\sec\theta\sqrt{\sec^2\theta-1}} = \int d\theta$.

The third distinct configuration of the right triangle (modulo interchanging the legs, of course) results in base angle θ, hypotenuse a, adjacent side x, and opposite side $\sqrt{a^2-x^2}$, and we use the substitutions $x = a\cos\theta$ and $dx = -a\sin\theta d\theta$. For example, $\int \frac{dx}{\sqrt{1-x^2}} = \int \frac{-\sin\theta d\theta dx}{\sqrt{1-x^2}} = -\int d\theta$.

Reverse trig substitutions:

Given a right triangle with base angle θ, adjacent side $\frac{1-u^2}{2}$ and opposite side u, we immediately have that the hypotenuse is $\sqrt{\left(\frac{1-u^2}{2}\right)^2+u^2} = \frac{1+u^2}{2}$. The fundamental trig ratios are then: $\sin\theta = \frac{2u}{1+u^2}$, $\cos\theta = \frac{1-u^2}{1+u^2}$, and $\tan\theta = \frac{2u}{1-u^2}$. Also

$$\frac{d\sin\theta}{du} = \cos\theta\frac{d\theta}{du} = \frac{2(1+u^2)-(2u)^2}{(1+u^2)^2},$$ so

$$\frac{d\theta}{du} = \frac{1}{\cos\theta}\left(\frac{2(1-u^2)}{(1+u^2)^2}\right) = \frac{1+u^2}{1-u^2}\left(\frac{2(1-u^2)}{(1+u^2)^2}\right) = \frac{2}{1+u^2},$$ then it is clear that $d\theta = \frac{2du}{1+u^2}$.

Now consider $\int \frac{d\theta}{2+\cos\theta}$. Making the substitutions from our theory, this integral becomes

$$\int \frac{1}{2+\frac{1-u^2}{1+u^2}}\frac{2du}{1+u^2} = 2\int \frac{du}{3+u^2} = \frac{2\sqrt{3}}{3}\int \frac{d\left(\frac{u}{\sqrt{3}}\right)}{1+\left(\frac{u}{\sqrt{3}}\right)^2}$$ which reduces to $\frac{2\sqrt{3}}{3}\arctan\frac{u}{\sqrt{3}}$.