Tabular Integration (by Parts)

Example: Find $\int x^4 e^x \, dx$

1) Decide which function in integrand is going to be differentiated - we choose x^4

2) Then the remaining part of the integrand will need to be integrated - that's e^x

3) Set up table:

d/dx	$\int dx$
$\vdots$	$\vdots$

4) Enter choices from Steps 1) and 2) :

d/dx	$\int dx$
x^4	e^x
$\vdots$	$\vdots$

5) Differentiate successively in column 1 and integrate successively in column 2 :

d/dx	$\int dx$
x^4	e^x
$4x^3$	e^x
$12x^2$	e^x
$24x$	e^x
24	e^x
0	e^x

STOP HERE!

6) Add diagonal lines and parities, starting with $\oplus$ and alternating with $\ominus$:

d/dx		$\int dx$
x^4	$+$	e^x
$4x^3$	$-$	e^x
$12x^2$	$+$	e^x
$24x$	$-$	e^x
24	$+$	e^x
0		e^x

...

7) Interpret solution by multiplying terms connected on diagonals, applying parity (+ or −), then adding all products:

$$\int x^4 e^x \, dx = x^4 e^x - 4x^3 e^x + 12x^2 e^x - 24x e^x + 24 e^x$$
$$= e^x \left[x^4 - 4x^3 + 12x^2 - 24x + 24 \right]$$

Done

8) Handling case where differentiation does not eventually yield zero:

Example: $\int e^x \sin x \, dx$

9) Set up table and repeat steps as before

d/dx	$\int dx$
$\sin x$	e^x
$\cos x$	e^x
$-\sin x$	e^x

stop when row has same terms as first row (ignore sign)

9)

d/dx	sign	$\int dx$
$\sin x$		e^x
	+	
$\cos x$		e^x
	−	
$-\sin x$		e^x

10) Add horizontal line in last row and assign + parity:

d/dx	sign	$\int dx$
$\sin x$		e^x
	+	
$\cos x$		e^x
	−	
$-\sin x$	+	e^x

11) Interpret table as before but write last row as integral:

$$\int e^x \sin x \, dx = \sin x \cdot e^x - \cos x \cdot e^x + \int e^x(-\sin x)\,dx$$

12) Move integral on right to other side:

$$2\int e^x \sin x\,dx = \sin x \cdot e^x - \cos x \cdot e^x$$

and simplify:

$$\int e^x \sin x\,dx = \frac{e^x}{2}(\sin x - \cos x)$$

13) Handling case where initial row does not re-appear. You can terminate the table at any time and treat the last row as an integral. In 10) we were lucky and saw the original problem integral appear. In this (hard) example, we keep adding rows until we see an integral of the last row that we (think we) can do:

Example: $\int x \arctan x \, dx$

If we choose to differentiate the x away, we are faced with integrating arctan x. So we go the other way and get rid of the inverse trig function:

d/dx		$\int dx$
$\arctan x$	$+$	x
$\frac{1}{1+x^2}$	$-$	$\frac{x^2}{2}$
$-\frac{2x}{(1+x^2)^2}$	$+$	$\frac{x^3}{6}$

So $\int x \arctan x = \frac{1}{2}x^2 \arctan x - \frac{1}{6}\frac{x^3}{1+x^2} +$

$$\left[-\frac{1}{3}\int \frac{x^4}{(1+x^2)^2}\,dx\right]$$

This last integral can be done by substitution and is:

$$\int \frac{x^4}{(1+x^2)^2}\,dx = x + \frac{1}{2}\frac{x}{x^2+1} - \frac{3}{2}\arctan x$$

So:

$$\int x \arctan x\,dx = \frac{1}{2}x^2 \arctan x - \frac{1}{6}\frac{x^3}{1+x^2} -$$

$$\frac{1}{3}\left(x + \frac{1}{2}\frac{x}{x^2+1} - \frac{3}{2}\arctan x\right)$$