FLOW CHART FOR INTEGRATION BY PARTIAL FRACTION DECOMPOSITION

Given $\int \frac{P(x)}{Q(x)} dx$ where $P(x)$ and $Q(x)$ are polynomials in x:

(1) Check the degrees of $P(x)$ and $Q(x)$. If $\deg P(x) \geq \deg Q(x) \geq 1$ then do the division. If $\deg P(x) < \deg Q(x)$ go to (3)

(a) If $\deg Q(x) = 1$, use synthetic division

(b) If $\deg Q(x) > 1$, use long division

(2) Split the quotient and remainder from (1). The quotient will be non-negative powers of x and easily integrated. Take the remainder $R(x)$, if any, to (3).

(3) We now have either $\frac{P(x)}{Q(x)}$ from (1) or $\frac{R(x)}{Q(x)}$ from (2). In either case the numerator degree is strictly less than the denominator degree. Factor $Q(x)$. This may be difficult, but every polynomial over the rationals may be expressed as a product of linear and/or quadratic factors with real (sometimes not even rational) coefficients. Usually, book problems are stated with the denominator already factored, but in practical problems, this is up to you. An example of a non-obvious factorization is $x^4 + 1 = \left(x^2 + \sqrt{2}\,x + 1\right)\left(x^2 - \sqrt{2}\,x + 1\right)$. You can always arrange for $Q(x)$ to be a monic polynomial, that is one with leading coefficient 1, by factoring out a suitable constant. For example, $2x^2 + 4x + 1 = 2\left(x^2 + 2x + \frac{1}{2}\right)$. Move the constant out of the integral.

(4) Suppose the factorization of $Q(x)$ is $(x+a_1)^{r_1}\cdots(x+a_n)^{r_n}(x^2+b_1x+c_1)^{s_1}\cdots(x^2+b_mx+c_m)^{s_m}$. So there are n linear factors, possibly raised to various powers, and m quadratic factors, also possibly raised to various powers. We are going to decompose the rational function from (3), either $\frac{P(x)}{Q(x)}$ from (1) or $\frac{R(x)}{Q(x)}$ from (2), into a sum of individual fractions.

(5) For every term of the form $(x+a)^r$ that appears in the factorization of $Q(x)$, we will write the following as part of the overall sum: $\frac{A_1}{(x+a)} + \frac{A_2}{(x+a)^2} + \cdots \frac{A_r}{(x+a)^r}$. Note that every power up to and including the r^{th} power is represented. If the factor $(x+a)$ is not repeated, then $r = 1$ and there is only one fraction written for this factor.

(6) For every term of the form $(x^2+bx+c)^s$ that appears in the factorization of $Q(x)$, we will write the following as part of the overall sum: $\frac{B_1x + C_1}{(x^2+bx+c)} + \frac{B_2x + C_2}{(x^2+bx+c)^2} + \cdots \frac{B_sx + C_s}{(x^2+bx+c)^s}$. Note that every power up to and including the s^{th} power is represented. If the factor (x^2+bx+c) is not repeated, then $s = 1$ and there is only one fraction written for this factor.

(7) Write out the overall sum (using $\frac{P(x)}{Q(x)}$ as an example):

$$\frac{P(x)}{Q(x)} = \frac{A_1}{(x+a)} + \frac{A_2}{(x+a)^2} + \cdots \frac{A_r}{(x+a)^r} + \cdots \frac{B_1x + C_1}{(x^2+bx+c)} + \frac{B_2x + C_2}{(x^2+bx+c)^2} + \cdots \frac{B_sx + C_s}{(x^2+bx+c)^s}$$

Clear fractions.

(8) Equate the coefficients on each side of the equation for each power of x separately (these powers are necessarily independent in that no amount of x^n can cancel x^m, for example).

(9) You now have a well-determined system of equations in the coefficients A_i, B_j, and C_k. You may solve the system, or...

(10) Very often using judicious numerical values of x in the results of (7) and (8) will allow you to determine the coefficients without much calculation. Try this first before you tackle the linear system.

(11) Rewrite the original integral as a sum of integrals of the appropriate fractions. The individual integrals should be power rule type (possibly logs) or arctangents after suitable substitutions, depending on how the coefficents work out.

AN EXAMPLE: $\int \frac{xdx}{x^3 - 1}$

(i) The degree of the numerator is 1 and the denominator 3, so we proceed to step (3).

(ii) $x^3 - 1 = (x-1)(x^2+x+1)$, note the second factor is irreducible, so steps (4) - (7)..

(iii) Set $\frac{x}{x^3-1} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}$

(iv) Clearing fractions,

$x = A(x^2+x+1) + (Bx+C)(x-1) = (A+B)x^2 + (A-B+C)x + (A-C)$

(v) Equating coefficients of powers $A+B = 0$, $A-B+C = 1$, and $A-C = 0$. There is your system of three equations in three unknowns.

(vi) Rather than tackle the system in (v), observe that if $x = 1$, $1 = 3A$, so $A = \frac{1}{3}$. Also, $A = C$, so $C = \frac{1}{3}$. Finally, since $A - B + C = 1$, we see $-B = \frac{1}{3}$, so $B = -\frac{1}{3}$.

(vii) $\int \frac{xdx}{x^3-1} = \frac{1}{3}\int \frac{dx}{x-1} + \frac{1}{3}\int \frac{(-x+1)dx}{x^2+x+1}$

(viii) You can practice the integrations by substitution

(*) btw, Maple has a parfrac command