DIFFERENTIAL GEOMETRY

Space Curves

Given position vector $\mathbf{r}(t)$ in Euclidean 3-space, we assume $\mathbf{r}'(t) \neq 0$ and $\mathbf{r}(t)$ is injective except possibly at its endpoints (so that it doesn't cross over itself or re-trace itself). You can think of t as time, and $\mathbf{r}(t)$ as a point in space tracing out the curve as time passes with velocity $\mathbf{r}'(t)$. Saying that $\mathbf{r}'(t) \neq 0$ is the same as saying the tracing never pauses. We also assume that the curve is smooth...which is a technical term that means $\mathbf{r}'(t)$ is a continuous function. This condition means that the tracing out process never reverses, either...because that would force $\mathbf{r}'(t) = 0$ somewhere.

Although it is convenient to think of t as representing time, it is an artificial notion, and we may want to develop a parametrization of the curve that is more natural. The natural choice for parametrizing space curves is the length of the curve (from some starting point which may be arbitrary). Denote the arc length parameter by s.

From the distance formula for space curves, we have $s(t) = \int_c^t \|\mathbf{r}'(\xi)\| d\xi$. This is an increasing function of t since the norm is strictly positive, and it follows that the arc length function is injective. We can then (at least in principle) invert it to find $t(s)$, and then we can compose it to find $r(t(s))$. By the Fundamental Theorem of Calculus (part one) we have $s'(t) = \|\mathbf{r}'(t)\|$ and since this norm is never zero, we have $t'(s) = (\|\mathbf{r}'(t)\|)^{-1}$.

One way to get a picture of what a space curve is like is to attach a local coordinate system to it at every point. Then the way this local system changes as the parameter increases (we move along the curve) gives insight regarding the shape of the curve. We will form a local coordinate system called the "moving trihedron" which will be a mobile cartesian system that follows the curve. It consists of three mutually orthogonal unit vectors: $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$.

The unit tangent vector $\mathbf{T}(s)$ is defined to be $\mathbf{r}'(s)$. It is not immediately obvious that the vector $\mathbf{r}'(s)$ has unit length, nor is it obvious that it shows the linearized trend, or tangent, to the curve.

$\mathbf{r}'(s) = \mathbf{r}'(t) \cdot t'(s) = \mathbf{r}'(t)/\|\mathbf{r}'(t)\|$, so we see it is a vector divided by its norm, i.e. normalized to 1.

$\mathbf{r}'(s) = \lim_{\Delta s \to 0}\left(\dfrac{\mathbf{r}(s+\Delta s) - \mathbf{r}(s)}{\Delta s}\right)$ which is the limiting value of the vector that connects two points along the curve divided by the arc length between them, as those points become arbitrarily close...i.e. the tangent.

The unit normal vector $\mathbf{N}(s)$ is defined to be the vector of unit length in the direction of change in the tangent vector $\mathbf{T}$. This would mean $\mathbf{N}(s) = \mathbf{T}'(s)/\|\mathbf{T}'(s)\|$. Remember, just because $\mathbf{T}(s)$ is a unit vector, its derivative doesn't have to be. It should be clear that the unit normal can't exist if the unit tangent vector is constant...in this case we are on a piece of "straight track". In general, though, the curve is twisting around in space and the moving trihedron is well-defined. The unit normal is perpendicular to the unit tangent because any vector that has constant length (but not direction) is perpendicular to its derivative. Why is this so?

Finally, the unit binormal vector $\mathbf{B}(s)$ is defined to be $\mathbf{T}(s) \times \mathbf{N}(s)$. It is perpendicular to the other two.

There are three reference planes that are used to talk about space curves...the osculating (kissing) plane is the plane containing $\mathbf{T}$ and $\mathbf{N}$. This plane is the one that instantaneously contains the curve. The normal plane contains $\mathbf{N}$ and $\mathbf{B}$. And the rectifying plane contains $\mathbf{T}$ and $\mathbf{B}$. The unit binormal is always perpendicular to the osculating plane and is the momentary axis about which the tangent adjusts in the direction of the unit normal. This tendency is measured by the scalar curvature, denoted by $\kappa = \|\mathbf{T}'(s)\|$. The reciprocal of the curvature is the radius of curvature, denoted by ρ. This is the radius of the osculating circle, which represents the circle that most nearly conforms to the curve in the osculating plane.

In non-planar curves, the osculating plane can rock back and forth as the curve is traced out. This means the direction of the normal (and the other two unit vectors) is not constant. The tendency of the normal to change direction as arc length increases, measured by its projection onto the binormal, is called the torsion of the curve. It is denoted by $\tau = \mathbf{N}'(s) \cdot \mathbf{B}$.

Frenet-Serret Formulæ

Expressions for the derivatives of the three unit vectors that make up the moving trihedron are called collectively the Frenet-Serret formulas.

The first is easy, because it is already a definition. Since $\mathbf{N}(s) = \mathbf{T}'(s)/\|\mathbf{T}'(s)\|$ we have immediately

$\mathbf{T}'(s) = \|\mathbf{T}'(s)\| \cdot \mathbf{N}(s)$ or since the scalar rate of change of the tangent vector with respect to arc length is precisely the curvature, $\mathbf{T}'(s) = \kappa\mathbf{N}(s)$.

The expression for $\mathbf{N}'(s)$ is not so simple. Any vector can be written as the linear combination of its projections onto three mutually orthogonal vectors. So
$\mathbf{N}' = (\mathbf{N}' \cdot \mathbf{T})\mathbf{T} + (\mathbf{N}' \cdot \mathbf{N})\mathbf{N} + (\mathbf{N}' \cdot \mathbf{B})\mathbf{B}$

Consider $\mathbf{N}' \cdot \mathbf{T}$. We have $0 = (\mathbf{N} \cdot \mathbf{T})' = \mathbf{N}' \cdot \mathbf{T} + \mathbf{N} \cdot \mathbf{T}'$, so
$\mathbf{N}' \cdot \mathbf{T} = -\mathbf{N} \cdot \mathbf{T}' = -\kappa(\mathbf{N} \cdot \mathbf{N}) = -\kappa$

Now consider $\mathbf{N}' \cdot \mathbf{N}$. $\|\mathbf{N}\|^2 = 1$, so $(\mathbf{N} \cdot \mathbf{N})' = (\|\mathbf{N}\|^2)' = 2\mathbf{N}' \cdot \mathbf{N} = 0$

Finally, we have $\mathbf{N}' \cdot \mathbf{B} = \tau$, by definition.

Putting everything together, $\mathbf{N}'(s) = -\kappa\mathbf{T}(s) + \tau\mathbf{B}(s)$.

$\mathbf{B}'(s)$ goes the same way. $\mathbf{B}' = (\mathbf{B}' \cdot \mathbf{T})\mathbf{T} + (\mathbf{B}' \cdot \mathbf{N})\mathbf{N} + (\mathbf{B}' \cdot \mathbf{B})\mathbf{B}$, so...

$0 = (\mathbf{B} \cdot \mathbf{T})' = \mathbf{B}' \cdot \mathbf{T} + \mathbf{B} \cdot \mathbf{T}' = \mathbf{B}' \cdot \mathbf{T} + (-\kappa\mathbf{N} \cdot \mathbf{T}) = \mathbf{B}' \cdot \mathbf{T} + 0$, so $\mathbf{B}' \cdot \mathbf{T} = 0$

$(\mathbf{B}' \cdot \mathbf{N}) = \tau$, by definition, and

$\mathbf{B}' \cdot \mathbf{B} = 0$ because $(\|\mathbf{B}\|^2)' = 0$.

Therefore, $\mathbf{B}'(s) = \tau \boldsymbol{N}$.

The Frenet-Serret formulas are based on being able to change the parameter t into the arc length parameter s. If this is a difficult matter, it would be nice to have the formulas directly in terms of t. So let's derive them.

Recall we started with $\mathbf{r}'(t) = \mathbf{r}'(s)s'(t) = \mathbf{T}(s)s'(t)$

Next we calculate

$\mathbf{r}''(t) = \mathbf{r}''(s)[s'(t)]^2 + \mathbf{r}'(s)s''(t) = \mathbf{T}'(s)[s'(t)]^2 + \mathbf{T}(s)s''(t) = \kappa\mathbf{N}(s)[s'(t)]^2 + \mathbf{T}(s)s''(t)$

Finally, we calculate $\mathbf{r}'''(t) = \mathbf{N}'(s)\kappa[s'(t)]^3 + \mathbf{N}(s)[\kappa[s'(t)]^2]' + \mathbf{T}'(s)s''(t)s'(t) + \mathbf{T}(s)s'''(t) =$

$[-\kappa\mathbf{T}(s) + \tau\mathbf{B}(s)][\kappa[s'(t)]^3] + \lambda_1\mathbf{T}(s) + \lambda_2\mathbf{N}(s) = \kappa\tau\mathbf{B}(s)[s'(t)]^3 + \lambda_3\mathbf{T}(s) + \lambda_2\mathbf{N}(s)$, where the $\lambda' s$ are undetermined coefficients (we could determine them, but they won't show up in our answers).

Now form $\mathbf{r}'(t) \times \mathbf{r}''(t) = \mathbf{T}(s)s'(t) \times [\kappa\mathbf{N}(s)[s'(t)]^2 + \mathbf{T}(s)s''(t)] =$

$\mathbf{T}(s)s'(t) \times \mathbf{T}(s)s''(t) + \mathbf{T}(s)/\|\mathbf{r}'(t)\| \times \kappa\mathbf{N}(s)[s'(t)]^2 =$

$[s''(t)s'(t)]\mathbf{T}(s) \times \mathbf{T}(s) + [\kappa[s'(t)]^2 s'(t)\mathbf{T}(s) \times \mathbf{N}(s) = \kappa[s'(t)]^3\mathbf{B}(s)$, since $\mathbf{T}(s) \times \mathbf{N}(s) = \mathbf{B}(s)$.
We conclude $\mathbf{B}(s) = \mathbf{r}'(t) \times \mathbf{r}''(t)/\kappa[s'(t)]^3$

Taking the norm of both sides, we conclude $\|\mathbf{r}'(t) \times \mathbf{r}''(t)\| = \kappa[s'(t)]^3 = \kappa\|\mathbf{r}'(t)\|^3$

Summarizing what we have, $\mathbf{r}'(t) = \mathbf{T}(s)s'(t)$ implies $\mathbf{T}(s) = \mathbf{r}'(t)/\|\mathbf{r}'(t)\|$

and $\kappa = \|\mathbf{r}'(t) \times \mathbf{r}''(t)\|/\|\mathbf{r}'(t)\|^3$, which allows us further to say that

$\mathbf{B}(s) = \mathbf{r}'(t) \times \mathbf{r}''(t)/\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|$

Since $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$ form a right handed orthonormal system, we have $\mathbf{N} = \mathbf{B} \times \mathbf{T}$

Finally, $(\mathbf{r}'(t) \times (\mathbf{r}''(t)) \cdot \mathbf{r}'''(t)) = \kappa[s'(t)]^3\mathbf{B}(s) \cdot \mathbf{r}'''(t)) = \kappa[s'(t)]^3\mathbf{B}(s) \cdot \kappa\tau\mathbf{B}(s)[s'(t)]^3 = \kappa^2\tau[s'(t)]^6$, so $\tau = (\mathbf{r}'(t) \times (\mathbf{r}''(t)) \cdot \mathbf{r}'''(t))/\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|^2$