FALL 24 - CALCULUS 3 - EXAM 4 - Solutions

All problems equal points. 60 minutes.

1) Find the volume of the figure that is above the xy-plane, inside the cylinder given by $x^2 + y^2 = 4$, and below the paraboloid given by $z = x^2 + y^2$.

We choose cylindrical coordinates for this problem. The floor of the figure is the circular cross section (radius 2) of the cylinder in the xy-plane The paraboloid and the cylinder intersect in a circle of radius 2 in the plane at $z = 4$. By the rotational symmetry of the figure about the z-axis, the limits of integration are $0 \le \theta \le 2\pi$ and $0 \le r \le 2$. The paraboloid is the "roof" of the figure, so $0 \le z \le x^2 + y^2 = r^2$. We have the volume integral:
$\int_{\theta=0}^{2\pi}\int_{r=0}^{2}\int_{z=0}^{r^2} rdzdrd\theta = \int_{\theta=0}^{2\pi}\int_{r=0}^{2}[z]_0^{r^2} rdrd\theta = \int_{\theta=0}^{2\pi}\int_{r=0}^{2} r^3drd\theta$*. This becomes*
$\int_{\theta=0}^{2\pi}\left[\frac{r^4}{4}\right]_0^2 d\theta = 4 \cdot 2\pi = 8\pi$*.*

2) A thin circular disk is centered at the origin and extends from $x = -2$ to $x = 2$. The (area) density of the disk is given by $\sigma(x,y) = e^{-a(x^2+y^2)}$ in grams per sq cm. What is the total mass of the disk in grams?

Switching to polar coordinates, we have a circle of radius 2 centered at the origin with density function $\sigma(r) = e^{-ar^2}$. Total mass $M = \int_{\theta=0}^{2\pi}\int_{r=0}^{2}\sigma(r)rdrd\theta = \int_{\theta=0}^{2\pi}\int_{r=0}^{2} e^{-ar^2}rdrd\theta$. For the r integral $\int_{r=0}^{2} e^{-ar^2}rdr$, let $u = -ar^2$, then $du = -2ardr$, so the integral becomes
$\int_{r=0}^{2} e^u\left(\frac{du}{-2a}\right) = \frac{-1}{2a}\int_{r=0}^{2} e^u du = \frac{-1}{2a}\left[e^{-ar^2}\right]_{r=0}^{2}$*. This evaluates to $\frac{-1}{2a}[e^{-4a} - 1]$. Since $a > 0$ for there to be any physical mass, we can rewrite this as $\frac{1-e^{-4a}}{2a}$. Finally,*
$M = \frac{1-e^{-4a}}{2a}\int_{\theta=0}^{2\pi} d\theta = \frac{\pi(1-e^{-4a})}{a}$.

3) Find the centroid of the figure bounded by the x-axis and the curve $y = \sqrt{a^2 - x^2}$.

This is the part of the disk of radius a centered at the origin in the upper half plane. Since it is symmetric about the y-axis, the coordinate $\bar{x} = 0$. The only remaining question is the value for $\bar{y}$. This is given by finding the moment of the half-disk area about the x-axis:
$M_x = \int_{x=-a}^{a}\int_{y=0}^{\sqrt{a^2-x^2}} ydydx$*. We have $M_x = \int_{x=-a}^{a}\left[\frac{y^2}{2}\right]_0^{\sqrt{a^2-x^2}} dx = \frac{1}{2}\int_{x=-a}^{a}(a^2 - x^2)dx$. This reduces to $M_x = \frac{1}{2}\left[a^2x - \frac{x^3}{3}\right]_{-a}^{a} = \frac{1}{2}\left[\left(a^3 - \frac{a^3}{3}\right) - \left(-a^3 + \frac{a^3}{3}\right)\right] = \frac{1}{2}\left(\frac{4a^3}{3}\right) = \frac{2a^3}{3}$. The area A of the half-disk is $\frac{\pi a^2}{2}$, so $\bar{y} = \frac{M_x}{A} = \frac{2a^3}{3} \cdot \frac{2}{\pi a^2} = \frac{4a}{3\pi}$. Then the centroid is at $\left(0, \frac{4a}{3\pi}\right)$.*

4) A quartz crystal occupies the space in the first octant where $0 \le x \le 1$, $0 \le z \le 1 - x$, and

$0 \le y \le 3 - x - z$. The internal temperature of the crystal is given by the function $T(x,y,z) = x$ in degrees celsius. What is the weighted average of the temperature over the entire crystal?

The volume integral is $\int_0^1\int_0^{1-x}\int_0^{3-x-z} dydzdx = \int_0^1\int_0^{1-x}(3-x-z)dzdx = \int_0^1\left[3z - xz - \frac{z^2}{2}\right]_0^{1-x}dx$*.* *This reduces to* $\int_0^1\left(3(1-x) - x(1-x) - \frac{(1-x)^2}{2}\right)dx = \frac{7}{6}$*. Now we need the temperature weighted integral:* $\int_0^1\int_0^{1-x}\int_0^{3-x-z} T(x,y,z)dydzdx = \int_0^1\int_0^{1-x}\int_0^{3-x-z} xdydzdx$*. This is similar to the volume integral and becomes* $\int_0^1\int_0^{1-x}\left(3x - \frac{x^2}{2} - zx\right)dzdx$*, which reduces to* $\int_0^1\left(3x(1-x) - x^2(1-x) - \frac{x(1-x)^2}{2}\right)dx = \frac{3}{8}$*. Finally, the average temperature in the crystal is* $\overline{T} = \frac{\frac{3}{8}}{\frac{7}{6}} = 0.32$ *deg C.*