FALL 2024 - CALCULUS 3 - TEST #3B - Solutions

1) Find the gradient of $f(x,y,z) = \sin(e^{x\cos(yz)})$

$f_x = \cos(yz)e^{x\cos(yz)} \cdot \cos(e^{x\cos(yz)})$
$f_y = -xz\sin(yz)e^{x\cos(yz)} \cdot \cos(e^{x\cos(yz)})$
$f_z = -xy\sin(yz)e^{x\cos(yz)} \cdot \cos(e^{x\cos(yz)})$
so the gradient $\nabla f = e^{x\cos(yz)} \cdot \cos(e^{x\cos(yz)})\left[\cos(yz)\hat{\mathbf{i}} - xz\sin(yz)\hat{\mathbf{j}} - xy\sin(yz)\hat{\mathbf{k}}\right]$

2) Find the equation of the tangent plane to the surface given by $z = x^2 + y^2$ at the point $(3,3,18)$

$\nabla(x^2 + y^2 - z) = 2x\hat{\mathbf{i}} + 2y\hat{\mathbf{j}} - k$. *Evaluated at* $(3,3,18)$*, this is* $6\hat{\mathbf{i}} + 6\hat{\mathbf{j}} - k$. *This is direction vector of plane* $6x + 6y - z = d$. *Since plane passes thru* $(3,3,18)$*, this means* $18 + 18 - 18 = d$*, or* $d = 18$. *Then the tangent plane has equation* $6x + 6y - z = 18$.

3) If $\phi(x,y) = e^x y^3 \sin(xy)$, find ϕ_{xy} and ϕ_{yx}

$\phi_x = e^x y^3 \sin(xy) + e^x y^4 \cos(xy)$
$\phi_{xy} = e^x[(3y^2 - xy^4)\sin(xy) + (xy^3 + 4y^3)\cos(xy)] = y^2 e^x[(3 - xy^2)\sin(xy) + (xy - 4y)\cos(xy)]$.
The function and its partials up to second order are continuous, so Clairaut's Theorem applies and $\phi_{xy} = \phi_{yx}$

4) Given $f(x,y,z) = \sin(x)e^{yz}$ with $x(t) = \cos t$, $y(t) = \tan t$, and $z(t) = e^{t^2}$, find $f'(t)$

Compose first, then differentiate: $f(t) = \sin(\cos t)\exp(e^{t^2}\tan t)$, then
$f'(t) = -\sin(t)\cos(\cos(t))\exp\left(\tan(t)e^{t^2}\right) + e^{t^2}\sin(\cos(t))\exp\left(\tan(t)e^{t^2}\right)[\sec^2(t) + 2t\tan(t)]$
This can be reduced further to
$\exp\left(\tan(t)e^{t^2}\right)\left[e^{t^2}\sin(\cos(t))(\sec^2(t) + 2t\tan(t)) - \sin(t)\cos(\cos(t))\right]$

or use $\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$, then $\frac{df}{dt} = \frac{\partial f}{\partial x}(-\sin t) + \frac{\partial f}{\partial y}(\sec^2 t) + \frac{\partial f}{\partial z}\left(e^{t^2} \cdot e^{e^{t^2}}\right)$
$\frac{\partial f}{\partial x} = \cos(x)e^{yz}$, $\frac{\partial f}{\partial y} = \sin(x)ze^{yz}$, and $\frac{\partial f}{\partial z} = \sin(x)ye^{yz}$
Putting this all together we get
$\frac{df}{dt} = \cos(x)e^{yz}(-\sin t) + \sin(x)ze^{yz}(\sec^2 t) + \sin(x)ye^{yz}\left(e^{t^2} \cdot e^{e^{t^2}}\right)$...much easier

5) Suppose $\phi = f(x,y,z,t)$, $x = x(u,t)$, $y = y(v,t)$, $z = z(u,v)$, $u = u(t)$, and $v = v(t)$. Draw the dependency diagram for $\phi'(t)$

See separate..