FALL 2024 - CALCULUS 3 - EXAM 2B - Solutions

1) For a cannon with muzzle velocity $\mathbf{v}_0$, find (with explanation) the launch angle yielding maximum range. Assume both launch and impact at ground level.
Let $R(\theta)$ be the range as a function of launch angle for a given v_0. Flight time is t_f. We have in the vertical direction $|\mathbf{v}_0|t_f\sin\theta - \frac{1}{2}gt_f^2 = 0$ at impact. Solving for t_f we get $t_f = \frac{2|\mathbf{v}_0|\sin\theta}{g}$. Then the range becomes $R(\theta) = |\mathbf{v}_0|t_f\cos\theta = |\mathbf{v}_0|\frac{2|\mathbf{v}_0|\sin\theta}{g}\cos\theta = \frac{|\mathbf{v}_0|^2\sin 2\theta}{g}$. Then $R'(\theta) = \frac{2|\mathbf{v}_0|\cos 2\theta}{g} = 0$ implies $\theta = 45^o$. Since $R''(45^o) < 0$, we see this value of θ maximizes the range for any given muzzle velocity.

2) The "Green Monster" at Fenway Park in Boston is the left field fence at the baseball park. The fence is 310 feet from home plate down the left field line. Although this distance is smaller than that in most major league parks, the fact that it is 37 feet high makes it difficult to hit a home run down the left field line. If a baseball is hit down the line starting at 3 feet off the ground and traveling at 110 mph initial velocity (which is constant since we are neglecting air resistance), what is the minimum angle of inclination from horizontal that would just clear the fence?
The trajectory equation is $\mathbf{r}(t) = (v_0\cos\theta)t\hat{\mathbf{i}} + ((v_0\sin\theta)t - 16t^2)\hat{\mathbf{j}}$. We do the problem in feet/sec. 110 miles per hour is 161.333 ft per second. Each component in the trajectory equation gives a relation between t and θ, but although we have two equations in two unknowns, it is a non-linear situation. We solve by eliminating t and obtain a trigonometric relation for θ. So...,since the fence is 310 ft from the point of launch, the time to reach the wall is given by $t_{wall} = \frac{310}{161.333\cos\theta} = \frac{1.921}{\cos\theta}$. Plugging this into the equation for height and setting it equal to 34 ft (we start 3 ft off the ground) we get
$(161.333\sin\theta)\left(\frac{1.921}{\cos\theta}\right) - 16\left(\frac{1.921}{\cos\theta}\right)^2 = 34$*. This simplifies to* $310\frac{\sin\theta}{\cos\theta} - \frac{59.044}{\cos^2\theta} = 34$*,* *which we write as* $310\tan\theta - 59.044\sec^2\theta = 34$, since $\frac{1}{\cos\theta} = \sec\theta$*. Using* $\sec^2\theta = 1 + \tan^2\theta$ *we recover a quadratic in* $\tan\theta$: $310\tan\theta - 59.044(1+\tan^2\theta) = 34$, or
$59.044\tan^2\theta - 310\tan\theta + 93.044 = 0$. *Using the quadratic formula we get roots* $\tan\theta = 0.3196$.and 4.931. These correspond to angles 17.72^o *and* 78.54^o*. The smaller angle corresponds to a flat trajectory that just makes it over the fence (line drive) and the larger angle corresponds to a high parabolic arc (moon shot) that, if it were any higher, would come down and hit the fence or the field before 310 feet. Any launch angle in between these angles would result in a home run.*

3) A metric machine screw is 5 mm long, has a diameter of 2 mm and a pitch of 0.4 mm. This is about the smallest screw that is commonly available at a hardware store. The diameter of a threaded cylinder like a screw is measured at the largest cross-section (not at the groove where it is indented). So the thread of the screw forms a helix of constant pitch 0.4 mm and diameter 2 mm. What is the length of the thread (helix) that makes up 80% of

the length of the screw?

Aligning the screw axis with the positive z*-axis and starting the helical path in the* xy*-plane, we can write the space curve equation as* $\mathbf{r}(t) = (\cos t)\hat{\mathbf{i}} + (\sin t)\hat{\mathbf{j}} + \left(\frac{0.40}{2\pi}t\right)\hat{\mathbf{k}}$*. Note that the radius is* 1 *mm, and as* t *increases by* 2π *the helix goes around once and rises* 0.4 *mm.* 80% *of* 5 *mm is* 4 *mm, so the helix has to go around* 10 *times, hence* t *goes from* 0 *to* 20π*. Distance along the path is* $s = \int_0^{20\pi} |\mathbf{r}'(t)| dt = \int_0^{20\pi} \sqrt{(-\sin t)^2 + (\cos t)^2 + \frac{0.16}{4\pi^2}}\, dt$*. This simplifies to* $\int_0^{20\pi} 1.0020 dt = 1.0020[t]_0^{20\pi} = 62.96$ *mm.*

4) An anti-aircraft gun is aimed due east and fires projectiles traveling at 1200 ft per sec. An enemy aircraft flying due west towards the location of the gun is traveling at 400 mph and constant altitude 5,000 feet. If the gun is inclined 60 degrees from horizontal, is it capable of shooting down the plane within 10 seconds? If so, how many miles east of the gun location could this occur and how long in seconds after the gun is fired? How far (line of sight distance) was the plane from the gun when the gun was fired?

The trajectory equation is $\mathbf{r}(t) = (1200\cos 60^o)t\hat{\mathbf{i}} + ((1200\sin 60^o)t - 16t^2)\hat{\mathbf{j}}$*. The anti-aircraft shell reaches the plane altitude whenever* $(1200\sin 60^o)t - 16t^2 = 5000$*. Rewriting this as a quadratic,* $16t^2 - 1039.2t + 5000 = 0$*. This has solutions* 5.233 *and* 59.717 *seconds. Obviously, the shorter time corresponds to hitting the enemy plane as the shell rises rather than falls. After* 5.233 *seconds, the shell is* $(1200)(0.500)(5.233) = 3139.8$ *feet (*0.595 *mi) east of the gun, so if the gun is at the origin, the plane is intercepted at the point* $(3139.8, 5000)$*. The plane is traveling at* 586.7 *feet per second, so when the gun was fired, the plane was* $(586.7)(5.233) = 3070.0$ *feet east of the point of interception. So that position was* $(6209.8, 5000)$*. This point is* 7972.6 *feet from the gun (origin) along the line of sight.*