FALL 2024 - CALCULUS 3 - EXAM 1B - Solutions

1) How far (perpendicular distance) is the point $(1,1,1)$ from the plane $3x - 5y + 7z = 12$?

First, draw a line segment from the given point $(1,1,1)$ *to some easily determined point in the plane. The point in the plane can be arbitrary since we are going to project the length of that line segment, viewed as a vector, onto a perpendicular to the plane. So we pick* $y = z = 0$*, then* $3x = 12$ *implies* $x = 4$*. Incidentally, this happens to be where the plane crosses the* x*-axis, but this fact is not essential to the determination of the distance. Now we have a line segment connecting* $(1,1,1)$ *and* $(4,0,0)$*. Taking the difference of corresponding coordinates, we convert it into the vector* $\langle -3,1,1 \rangle$*. This is the vector that would take you from* $(4,0,0)$ *to* $(1,1,1)$*. Just to provide a check on our final answer, let's find the length of this vector. It is (by the distance formula)* $\sqrt{(1-4)^2 + (1-0)^2 + (1-0)^2} = \sqrt{11}$*. Our final answer should be no greater than this distance, since (unless we were really lucky picking the point on the plane) we are looking for only the component of* $\langle -3,1,1 \rangle$ *perpendicular to the plane. In general, we expect* $\langle -3,1,1 \rangle$ *to be oblique to the plane. Next, we determine a direction vector for the plane...this is easy, we read off the coefficients from the given equation for the plane:* $\langle 3,-5,7 \rangle$*. It is only the direction of this vector that we need, so we unitize it:* $\mathbf{e}_\perp = \dfrac{\langle 3,-5,7 \rangle}{\sqrt{9+25+49}} = \dfrac{1}{9.11}\langle 3,-5,7 \rangle$*. We are ready to dot the line segment vector with the unit direction vector to get the distance from point to plane:* $\dfrac{1}{9.11}\langle 3,-5,7 \rangle \bullet \langle -3,1,1 \rangle = \dfrac{1}{9.11}(-9-5+7) = \frac{-7}{9.11} = -0.768$ *units. The negative sign indicates our two dotted vectors formed an obtuse angle between them. We are interested in the positive distance, so we take the absolute value to obtain a point/plane separation of* 0.768 *units.*

2) What is the (i) area of the triangle formed by the points $(4,7,-5)$, $(-6,5,1)$, and $(1,-2,-2)$?

(ii) is this a right triangle?

(*i) We are going to use the area formula based on the cross product, so we need to create some relevant vectors by choosing one point to be common to the two vectors going to the other points. So pick* $(1,-2,-2)$ *as the common point. Then the vector* $\mathbf{A} = \langle 3,9,-3 \rangle$ *goes to* $(4,7,-5)$ *and* $\mathbf{B} = \langle -7,7,3 \rangle$ *goes to* $(-6,5,1)$*. So the area is*

$$\frac{1}{2}|\mathbf{A}\times\mathbf{B}| = \frac{1}{2}\left\| \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 9 & -3 \\ -7 & 7 & 3 \end{vmatrix} \right\| = \frac{1}{2}|[48\mathbf{i} - (-12)\mathbf{j} + 84\mathbf{k}]| = \frac{1}{2}\sqrt{2304+144+7056} =$$

$\frac{1}{2}(97.48) = 48.74$ *units. The outside bars are for absolute value, the inside bars are for the determinant.*

(ii) If this triangle is a right triangle, it would have to satisfy the Pythagorean Theorem. So we check: $|\mathbf{A}| = 9.94$, $|\mathbf{B}| = 10.34$, *and the distance from* $(4,7,-5)$ *to* $(-6,5,1)$ *is* 11.83*. The squares are respectively* 99, 107, *and* 140*. There is no way these can be fitted together to satisfy the Pythagorean Theorem, so the triangle cannot have a 90 degree angle*.

3) What is the dihedral angle between the xz plane and the plane defined by the three noncollinear points $(1,2,3)$, $(3,5,7)$, and $(-2,-2,-4)$?

We need the angle between two direction vectors. The xz plane has direction vector $-\mathbf{j}$. We'll have to get the other one by constructing vectors from the given points. Using $(1,2,3)$ as the common point, we can write $\mathbf{A} = \langle 2,3,4\rangle$ and $\mathbf{B} = \langle -3,-4,-7\rangle$. Then

$\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 4 \\ -3 & -4 & -7 \end{vmatrix} = -5\mathbf{i} + 2\mathbf{j} + \mathbf{k} = \mathbf{C}$. *This vector is perpendicular to the plane with the three given points. The dihedral angle is the one between $-\mathbf{j}$ and $\mathbf{C}$. So* $\theta = \arccos\dfrac{-2}{(1)(\sqrt{30})} = 111.4^o$, *which we reduce to the supplementary angle* 68.6^o.

4) What is the distance from the line $\langle 1,1,4\rangle + t\langle 2,-2,5\rangle$ where $-\infty < t < \infty$ and the origin?

We need an arbitrary vector from the line to the origin (S). Set $t = 0$ and get point $P = (1,1,4)$. The vector $\overrightarrow{PS}$ becomes $\langle -1,-1,-4\rangle$. The direction vector for the line is $v = \langle 2,-2,5\rangle$. Its length is $\sqrt{4+4+25} = \sqrt{33}$. By our formula, the distance from the origin to the line is $\left|\dfrac{\overrightarrow{PS}\times\mathbf{v}}{\sqrt{33}}\right| = \dfrac{1}{\sqrt{33}}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & -1 & -4 \\ 2 & -2 & 5 \end{vmatrix} = \dfrac{1}{\sqrt{33}}|\langle -13,-3,4\rangle| = 2.42$ *units*

5) What is the angle between the line that connects $(1,2,2)$ to $(2,3,6)$ and the line that connects $(-5,-3,4)$ to $(2,3,6)$?

These lines meet at $(2,3,6)$, so we can create vectors and find the angle between them the usual way. Let $\mathbf{A} = \langle -1,-1,-4\rangle$ and $\mathbf{B} = \langle -7,-6,-2\rangle$. *Then* $\theta = \arccos\dfrac{21}{\sqrt{18}\sqrt{89}} = 58.3^o$.