SPRING 2026 - CALCULUS 3 - EXAM 1B - Solutions

1) If a particle were traveling on the path $\mathbf{r}(t) = \langle t, t^2, \cos 2\pi t\rangle$ and at $t = 5$ it suddenly went in a straight line at constant speed tangent to $\mathbf{r}(t)$, where would the particle be at $t = 10$?

At $t = 5$ the particle is at $r(5) = \langle 5, 25, \cos 10\pi\rangle = \langle 5, 25, 1\rangle$. Its velocity in general is $r'(t) = \langle 1, 2t, -2\pi \sin 2\pi t\rangle$, so at $t = 5$ we have $r'(5) = \langle 1, 10, 0\rangle$. Now it moves for five more seconds at this velocity which adds the distance vector $5 \cdot r'(5) = \langle 5, 50, 0\rangle$ to its starting position at $\langle 5, 25, 1\rangle$. The result is $\langle 5, 25, 1\rangle + \langle 5, 50, 0\rangle = \langle 10, 75, 1\rangle$, which is its position at $t = 10$.

2) What is the (i) area of the triangle formed by the points $(4, 7, -5)$, $(-6, 5, 1)$, and $(1, -2, -2)$?

(ii) is this a right triangle?

(*i) We are going to use the area formula based on the cross product, so we need to create some relevant vectors by choosing one point to be common to the two vectors going to the other points. So pick $(1, -2, -2)$ as the common point. Then the vector $\mathbf{A} = \langle 3, 9, -3\rangle$ goes to $(4, 7, -5)$ and $\mathbf{B} = \langle -7, 7, 3\rangle$ goes to $(-6, 5, 1)$. So the area is*

$$\frac{1}{2}|\mathbf{A}\times\mathbf{B}| = \frac{1}{2}\left\| \begin{matrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 9 & -3 \\ -7 & 7 & 3 \end{matrix} \right\| = \frac{1}{2}|[48\mathbf{i} - (-12)\mathbf{j} + 84\mathbf{k}]| = \frac{1}{2}\sqrt{2304 + 144 + 7056} =$$

$\frac{1}{2}(97.48) = 48.74$ *units. The outside bars are for absolute value, the inside bars are for the determinant.*

(ii) If this triangle is a right triangle, it would have to satisfy the Pythagorean Theorem. So we check: $|\mathbf{A}| = 9.94$, $|\mathbf{B}| = 10.34$, *and the distance from $(4, 7, -5)$ to $(-6, 5, 1)$ is 11.83. The squares are respectively* 99, 107, *and* 140. *There is no way these can be fitted together to satisfy the Pythagorean Theorem, so the triangle cannot have a 90 degree angle*.

3) What is the dihedral angle between the xz plane and the plane defined by the three noncollinear points $(1, 2, 3)$, $(3, 5, 7)$, and $(-2, -2, -4)$?

We need the angle between two direction vectors. The xz plane has direction vector $-\mathbf{j}$. We'll have to get the other one by constructing vectors from the given points. Using $(1, 2, 3)$ as the common point, we can write $\mathbf{A} = \langle 2, 3, 4\rangle$ and $\mathbf{B} = \langle -3, -4, -7\rangle$. Then

$$\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 4 \\ -3 & -4 & -7 \end{vmatrix} = -5\mathbf{i} + 2\mathbf{j} + \mathbf{k} = \mathbf{C}.$$

This vector is perpendicular to the plane with the three given points. The dihedral angle is the one between $-\mathbf{j}$ and $\mathbf{C}$. So

$\theta = \arccos \frac{-2}{(1)(\sqrt{30})} = 111.4^o$, *which we reduce to the supplementary angle* 68.6^o.

4) What is the angle between the line that connects $(1,2,2)$ to $(2,3,6)$ and the line that connects $(-5,-3,4)$ to $(2,3,6)$?

These lines meet at $(2,3,6)$, *so we can create vectors and find the angle between them the usual way. Let* $\mathbf{A} = \langle -1,-1,-4 \rangle$ and $\mathbf{B} = \langle -7,-6,-2 \rangle$. *Then* $\theta = \arccos \frac{21}{\sqrt{18}\sqrt{89}} = 58.3^o$.

5) For a cannon with muzzle velocity $\mathbf{v}_0$, find (with explanation) the launch angle yielding maximum range. Assume both launch and impact are at ground level.

From the book formula for range as a function of launch angle we have $R(\theta) = \frac{v_0^2}{g} \sin 2\theta$, *where* $0 \le \theta \le \frac{\pi}{2}$. *Differentiate this with respect to* θ *and set that equal to zero in order to find the extremum. So* $R'(\theta) = \frac{v_0^2}{g}(2\cos 2\theta) = 0$. *Solving for* $\cos 2\theta = 0$ *we find* $\theta = \frac{\pi}{4}$. *Technically, we should also calculate* $R''(\theta) = \frac{v_0^2}{g}(-4\sin\theta)$ *and set* $\theta = \frac{\pi}{4}$. *We find* $R''(\theta) = -4\frac{v_0^2}{g}\left(\sin\frac{\pi}{4}\right) < 0$, *which verifies we have a maximum range.*