ABSTRACT ALGEBRA - SPRING 2025 - EXAM 1B - Solutions

1) Show that $4x^2+6x+3$ is a unit in $\mathbb{Z}_8[x]$ and find its inverse.

Suppose $(ax^2+bx+c)(4x^2+6x+3) = 1 \bmod 8$*. Then* $4ax^4+(6a+4b)x^3+(3a+6b+4c)x^2+(3b+6c)x+3c = 1 \bmod 8$ *If* $c=3$*, then* $3c = 1 \bmod 8$*. If* $b=2$*, then* $3b+18 = 0 \bmod 8$*. If* $a=0$*, then* $3a+24 = 0 \bmod 8$*,* $6 \cdot 0 + 4 \cdot 2 = 0 \bmod 8$*, and* $4 \cdot 0 = 0 \bmod 8$*. We conclude* $(2x+3)(4x^2+6x+3) = 1 \bmod 8$*, so* $4x^2+6x+3$ *is a unit with inverse* $(2x+3)$*.*

2) Find all maximal ideals in $\mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$

$\mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$ *is not a field (there are zero divisors), so we expect ideals other than the trivial and improper ideals. Consider these three subrings of* $\mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$*:* $\{0\} \oplus \mathbb{R} \oplus \mathbb{R}$*,* $\mathbb{R} \oplus \{0\} \oplus \mathbb{R}$*, and* $\mathbb{R} \oplus \mathbb{R} \oplus \{0\}$*. They are clearly absorbing in each coordinate and they are all proper subrings of* $\mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$*.* $\{0\} \oplus \{0\} \oplus \mathbb{R}$ *is proper and absorbing, but strictly contained in both* $\{0\} \oplus \mathbb{R} \oplus \mathbb{R}$ *and* $\mathbb{R} \oplus \{0\} \oplus \mathbb{R}$*. Likewise for the other similar cases. Any other subring of* $\mathbb{R}$ *cannot be absorbing since the only ideals of* $\mathbb{R}$ *are* $\{0\}$ *and* $\mathbb{R}$*. So these are the only maximal ideals of* $\mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$*.*

3) Show that $\{0,2,4,6,8,10,12\}$ is a field modulo 14. What is the identity?

This is a subring of $\mathbb{Z}_{14}$ *so all the required properties of operations are present. We only need to show that each nonzero element is invertible. Evidently* 8 *is the identity:* $2 \times 8 = 16 = 2 \bmod 14$, $4 \times 8 = 32 = 4 \bmod 14$, $6 \times 8 = 48 = 6 \bmod 14$, $8 \times 8 = 64 = 8 \bmod 14$, $10 \times 8 = 80 = 10 \bmod 14$*, and* $12 \times 8 = 96 = 12 \bmod 14$*. For invertibility:* $2 \times 4 = 8 \bmod 14$, $6 \times 6 = 36 = 8 \bmod 14$, $8 \times 8 = 64 = 8 \bmod 14$, $10 \times 12 = 120 = 8 \bmod 14$.

4) Suppose that $u, v,$ and $u+v$ are units in a commutative ring. Show $u^{-1}+v^{-1}$ is also a unit.

Consider $(u+v)uv(u^{-1}+v^{-1}) = (u+v)(u^{-1}uv+v^{-1}uv) = (u+v)(v+u) = 1$*, since* $(u+v)$ *is a unit. Evidently* $(u^{-1}+v^{-1})^{-1} = (u+v)uv = (u^2v+uv^2)$*.*

5) Let R be a commutative unital ring. Let I be a proper ideal of R with the property that every element in $R - I$ is a unit. Show I is a unique maximal ideal of R.

Let $J \subseteq R$ *be any ideal that contains any element* x *other than those in* I*. Then* x *is invertible* *and this forces $J = R$. Then I must be maximal and also unique.*