FALL 2026 - CALCULUS 2 - TEST #3B - Solutions

1) (partial fractions) $\int \frac{dx}{x^2(x^2-4)}$

Note that $\frac{1}{x^2(x^2-4)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{(x-2)} + \frac{D}{(x+2)}$. Clearing fractions, $1 = Ax(x^2-4) + B(x^2-4) + Cx^2(x+2) + Dx^2(x-2)$. Collecting powers of x: $x^3(A+C+D) + x^2(B+2C-2D) + x(-4A+2C-2D) + 1(-4B)$. It is immediate that $-4B = 1$, so $B = -\frac{1}{4}$. We use trial values of x instead of setting up a system of linear equations. Setting $x = 2$ the only surviving term is $Cx^2(x+2) = C(4)(4) = 16C = 1$, so $C = \frac{1}{16}$. Not the coefficient of x^2, namely $(B+2C-2D)$ must be zero, so $(-\frac{1}{4} + 2 \cdot \frac{1}{16} - 2D) = 0$, or $\left(-\frac{1}{8} - 2D\right) = 0$, which implies $D = -\frac{1}{16}$. Finally, from the coefficient of x^3, we have $(A+C+D) = 0$ or $\left(A + \frac{1}{16} - \frac{1}{16}\right) = 0$, so $A = 0$. Now we have $\int \frac{dx}{x^2(x^2-4)} = -\frac{1}{4}\int \frac{dx}{x^2} + \frac{1}{16}\int \frac{dx}{(x-2)} - \frac{1}{16}\int \frac{dx}{(x+2)}$. These integrals are usually simple. $\int \frac{dx}{x^2(x^2-4)} = -\frac{1}{4}\left[-\frac{1}{x}\right] + \frac{1}{16}\ln|x-2| - \frac{1}{16}\ln|x+2|$. Simplified this is $\frac{1}{4x} + \frac{1}{16}\ln\left|\frac{x-2}{x+2}\right|$.

2) (trig substitution) $\int \frac{e^t dt}{\sqrt{9+e^{2t}}}$ (hard problem - do last and budget time)

Using the hints: Let $x = e^t$, then $dx = e^t dt$. $\int \frac{e^t dt}{\sqrt{9+e^{2t}}} = \int \frac{dx}{\sqrt{9+x^2}}$. Now let $x = 3\tan\theta$, then $dx = 3\sec^2\theta d\theta$, and $\int \frac{dx}{\sqrt{9+x^2}} = \int \frac{3\sec^2\theta d\theta}{\sqrt{9+9\tan^2\theta}}$. Factor out constants and use $1+\tan^2\theta = \sec^2\theta$. Then $\int \frac{3\sec^2\theta d\theta}{\sqrt{9+9\tan^2\theta}} = \int \frac{\sec^2\theta d\theta}{\sqrt{1+\tan^2\theta}} = \int \frac{\sec^2\theta d\theta}{\sqrt{\sec^2\theta}} = \int \sec\theta d\theta =$ (from hint) $\ln|\sec\theta + \tan\theta|$. Now $\tan\theta = \frac{x}{3}$, so $\sec\theta = \sqrt{1+\left(\frac{x}{3}\right)^2} = \frac{1}{3}\sqrt{9+x^2}$, so by back substitution twice we finally have $\int \frac{e^t dt}{\sqrt{9+e^{2t}}} = \ln\left|\frac{1}{3}\sqrt{9+x^2} + \frac{x}{3}\right| = \ln\left|\frac{1}{3}\left(e^t + \sqrt{9+e^{2t}}\right)\right| + C$

3) (integration by parts) $\int x^4 \cos 3x dx$ (obviously a tabular problem)

d/dx	$\int dx$
x^4	$\cos 3x$
$4x^3$	$\frac{\sin 3x}{3}$
$12x^2$	$\frac{-\cos 3x}{9}$
$24x$	$\frac{-\sin 3x}{27}$
24	$\frac{\cos 3x}{81}$
0	$\frac{\sin 3x}{243}$

, so after applying parity factors, the answer is

$$\frac{x^4 \sin 3x}{3} + \frac{4x^3 \cos 3x}{9} - \frac{12x^2 \sin 3x}{27} - \frac{24x \cos 3x}{81} + \frac{24 \sin 3x}{243} + C$$

4) (trigonometric) $\int \tan^4 x dx$

$$\int \tan^4 x dx = \int \tan^2 x(\sec^2 x - 1)dx = \int \tan^2 x \sec^2 x dx - \int \tan^2 x dx = \int \tan^2 x d(\tan x) - \int (\sec^2 x - 1)dx = \frac{1}{3}$$