# FALL 2026 - ABSTRACT ALGEBRA 1 - TEST 3B - Solutions

One reference sheet allowed.

1) If $\phi : G \to H$ is a homomorphism, show that $|\ker\phi| = 1$ implies $\phi$ is injective

If $|\ker\phi| = 1$, $\ker\phi = e_G$. Suppose $\phi(x) = \phi(y)$ for $x, y \in G$. Then $\phi(x)[\phi(y)]^{-1} = e_H$. It follows that $\phi(x)[\phi(y^{-1})] = \phi(xy^{-1}) = e_H$. Then we have $xy^{-1} \in \ker\phi$, which means $xy^{-1} = e_G$, or $x = y$. This establishes injectivity.

2) If $G$ is non-abelian and has order $p \cdot q$, where both $p$ and $q$ are prime, show that $|Z(G)| = 1$.

If $|Z(G)| \neq 1$, then by Lagrange's Theorem $|Z(G)| = p, q,$ or $pq$. It can't be $pq$ or $G$ would be abelian. Suppose it is $p$. Then $|G/Z| = q$, which implies $G/Z$ is cyclic, hence $G$ is abelian by the $G/Z$ theorem. This is a contradiction. Likewise, if $|G/Z| = p$. We conclude $|Z(G)| = 1$.

3) Show that the kernel of a homomorphism is always normal

Given $\phi : G \to H$ we want to show that for any $g \in G$, $g\ker\phi g^{-1} \in \ker\phi$. Using operation preservation, $\phi(g\ker\phi g^{-1}) = \phi(g)\phi(\ker\phi)\phi(g^{-1})$. This reduces to $\phi(g)e_H\phi(g^{-1}) = \phi(g)\phi(g^{-1}) = \phi(g)\phi(g)^{-1} = e_H$. It follows that $g\ker\phi g^{-1} \in \ker\phi$. Hence $\ker\phi \triangleleft G$.

4) Show that if two groups are isomorphic, they have exactly the same number of elements of each possible order

Suppose $G \cong H$. If $g \in G$ has order $n$, then $g^n = e_G$. By operation preservation, $\phi(g^n) = [\phi(g)]^n = \phi(e_G) = e_H$. So $h = \phi(g)$ evidently has order $n$ as well. IOW, for every element of order $n$ in $G$, there is a matching element in $H$ with the same order. Reversing the argument shows the claim is true.

5) Show that $A_n \neq Z(S_n)$ for $n \geq 3$

Since $[S_n : A_n] = 2$, $A_n \triangleleft S_n$. Then $S_n/A_n \cong \mathbb{Z}_2$, which is cyclic. Then by the $G/Z$ theorem, if $A_n = Z(S_n)$, $S_n$ would be abelian. None of the groups $S_n$ are abelian for $n \geq 3$. The contradiction establishes the claim.