FALL 2026 - ABSTRACT ALGEBRA 1 - TEST 1B - Solutions

No references. Questions 1-4 worth 25 points each plus bonus question worth 25 points.

1) State the axioms for an abelian group $\langle G, *\rangle$. Use appropriate mathematical symbolization, not a verbal description.

Given a set G with binary operation $*$, the following conditions must hold for an abelian group:

(i) $\forall a,b,c \in G,\ a*(b*c) = (a*b)*c$ *associativity*

(ii) $\exists! e \in G$ such that $\forall g \in G,\ e*g = g*e = g$ *identity*

(iii) For each $g \in G$, $\exists! g^{-1}$ such that $g*g^{-1} = g^{-1}*g = e$ *inverses*

(iv) $\forall a,b \in G,\ a*b = b*a$ *commutativity*

2) Suppose $\langle G, *\rangle$ is a group and $H,K \leq G$. Prove $H \cap K \leq G$.

Proof: Let $a,b \in H \cap K$. $H \cap K \neq \emptyset$ because $e_G \in H,K$. Note that $a,b \in H,K$, so $ab^{-1} \in H,K$, and it follows that $ab^{-1} \in H \cap K$. By the 1-step subgroup test $H \cap K \leq G$ ■

3) Suppose $\langle G, *\rangle$ is a group and $g \in G$ has order n. Show g^{-1} also has order n.

Given that $g^n = e$ where $n \in \mathbb{N}$ is minimal, multiply both sides by g^{-1} n times. Certainly $g^n * (g^{-1})^n = e = (g^{-1})^n$. At no point in this successive multiplication does $g^n * (g^{-1})^k = g^{n-k} = e$, or that would contradict the definition of the order of g. Apparently n is the minimal exponent that renders $(g^{-1})^n = e$, hence that is the order of g^{-1} also. ■

4) Show that for any group $\langle G, *\rangle$ with $a,b \in G$, $|ab| = |ba|$.

Suppose $|ab| = n$. Then $ababab\cdots ab = e = a(babab\cdots a)b$. Then $a(ba)^{n-1}b = e$, so $(ba)^{n-1} = a^{-1}b^{-1}$. This implies $(ba)^{n-1} = (ba)^{-1}$, but that means $(ba)^n = e$. So $|ba| = n$. ■

5) (Bonus) Write out the Cayley table for D_6

see book