Th^m: Compact metric spaces are complete

Pf: Let $\langle p_n \rangle$ be a cauchy sequence in X, compact metric space. Show it converges.

For $N = 1, 2, 3, \cdots$ define $E_N = \{p_N, p_{N+1}, \cdots\}$

Note $\lim_{N \to \infty} \operatorname{diam} \bar{E}_N = 0$ because of cauchy property & $\operatorname{diam} \bar{E} = \operatorname{diam} E$.

Each $\bar{E}$ is closed, hence compact.

By construction, $\bar{E}_N \supset \bar{E}_{N+1}$

Now by nested compact set theorem (3.10 b) $\exists!\, p \in X$ such that $p \in \bar{E}_N$, $\forall N \in \mathbb{N}$.

Fix $\varepsilon > 0$. $\exists N_0$ such that $\operatorname{diam} \bar{E}_N < \varepsilon$ whenever $N > N_0$. Note $p \in \bar{E}_N$, so $d(p, q) < \varepsilon$ for every $q \in \bar{E}_N$. Extend this to every $q \in E_N$.

We conclude $d(p, p_n) < \varepsilon$ if $n > N_0$,

or equivalently, $p_n \to p \in X$.

Cor... $\langle x_n \rangle$ cauchy sequence in $\mathbb{R}^k$. Want to

show bounded, since all bounded subsets

of $\mathbb{R}^k$ have compact closure (H-B).

Define $E_N = \{x_N, x_{N+1}, \dots\}$. As above

$\lim_{N \to \infty} \operatorname{diam} \overline{E}_N = 0$, so for some N_1,

$\operatorname{diam} E_{N_1} < 1$. Then the range of x_n

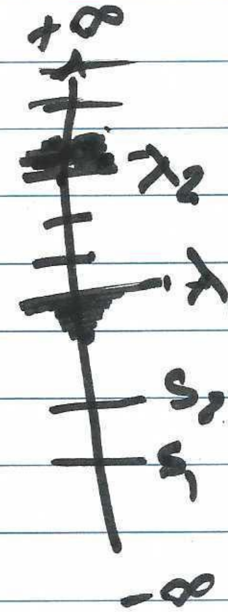
is range of $\{x_1, x_2, \cdots, x_{N-1}\}$ is A, so

the range of full sequence is bounded

by $A+1$. By H-B, $\overline{\{x_1, x_2, \cdots\}}$ compact

in $\mathbb{R}^k$.

Given sequence $\langle s_n \rangle$, let E be the set of all subsequential limits



Define $s^* = \sup E$

$s_* = \inf E$

$$s^* = \limsup_{n \to \infty} E$$

$$s_* = \liminf_{n \to \infty} E$$

limsup = limes superior
liminf = limes inferior

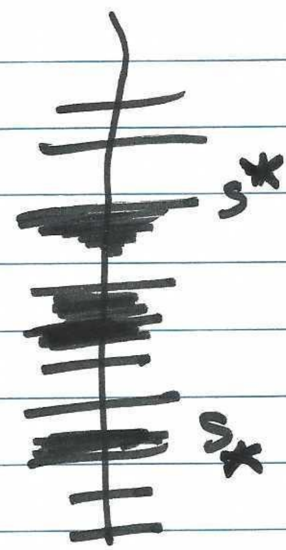
If $x > \limsup_n s_n$ then

Upper / Lower Limits

$s^* = \sup E$ ← all subseq limits

$s^* = \limsup_{n\to\infty} \{s_n\}$

$s_* = \liminf_{n\to\infty} \{s_n\}$



If $x > s^*$, only finitely many $s_n > x$

$x < s_*$ " " " $s_n < x$

Time out

$\frac{1}{n} \cdot \ln p = 0$

$n > N$ s.t. $\left|\frac{1}{n}\right| |\ln p| < \varepsilon$

$\frac{1}{n} < \frac{\varepsilon}{|\ln p|}$

$n > \frac{|\ln p|}{\varepsilon}$

$1 + n x_n \leq (1 + x_n)^n = p$

$n x_n = p - 1$

$0 \leq x_n \leq \frac{p-1}{n}$

Show $\sqrt[n]{n} \to 1$ as $n \to \infty$

3.20 Th^{m}

(a) $p>0$ $\quad \lim_{n\to\infty} \frac{1}{n^p} = 0$

Fix $\varepsilon>0$. Want to find $N\in\mathbb{N}$ s.t.

$$n>N \Rightarrow \frac{1}{n^p}<\varepsilon$$

$$n^p > \frac{1}{\varepsilon} \Rightarrow p\ln n > -\ln\varepsilon$$

$$\ln n > -\frac{\ln \varepsilon}{p}$$

$$n > \sqrt[p]{1/\varepsilon} \qquad n > e^{-\frac{\ln\varepsilon}{p}}$$

(b) If $p>0$, then $\lim_{n\to\infty} \sqrt[n]{p} = 1$

Case 1) $p>1$, then define $x_n = \sqrt[n]{p}-1$

Recall binomial th^{m}.

$$(x+1)^n = x^n + nx^{n-1} + \binom{n}{2}x^{n-2} + \cdots$$