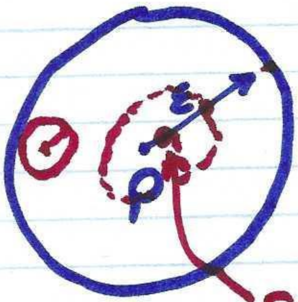


$\leftarrow B(p,\varepsilon)$

in $(0, \varepsilon/3) \ni r \in \mathbb{Q}$

$B(r,q) \subset B(p,\varepsilon)$

Given set E, define sequence in E

$\underline{a_1, a_2, \ldots a_k} \cdots \in E$

Sequence is a map: $\sigma : \mathbb{N} \to E$

σ: $a_1 \;\; a_2 \;\; a_3 \;\; a_4 \;\; a_5 \; \cdots \; a_k \;\; a_{k+1} \; \cdots$

$\mathbb{N} \to \mathbb{N}$ monotone

$a_1' \quad a_2' \quad a_3'$

$\Sigma : \mathbb{N} \times \mathbb{N} \to E$

$a_{n_1}, a_{n_2}, \ldots \quad n_2 > n_1$

Convergence of sequence in metric space

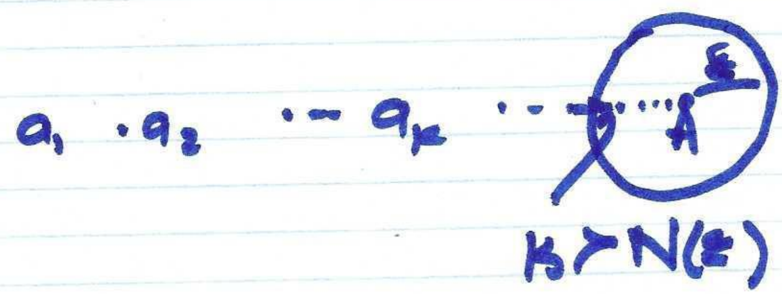
$$a_1, a_2, a_3, \cdots \to A$$

$\langle a_i \rangle \to A$ or $\lim_{i \to \infty} a_i = A$

If.. given any preassigned $\varepsilon > 0$,

$\exists\, N(\varepsilon) \in \mathbb{N}$ such that $n > N(\varepsilon)$

then $a_n \in B_d(A, \varepsilon)$



Th^m In metric spaces, limits of sequences are unique

Pf: FSOC suppose $\left.\begin{array}{l} \langle a_i \rangle \to A \\ \langle a_i \rangle \to B \end{array}\right\} A \neq B$

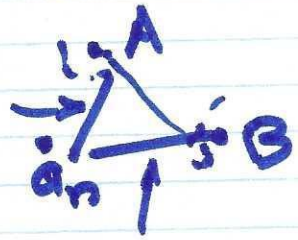
Since $\langle a_i \rangle \to A$ $\exists N(A,\varepsilon) \ni n > N(A,\varepsilon)$

$\Rightarrow a_n \in B(A,\varepsilon)$

$\langle a_i \rangle \to B$ $\exists N(B,\varepsilon) \ni n > N(B,\varepsilon)$

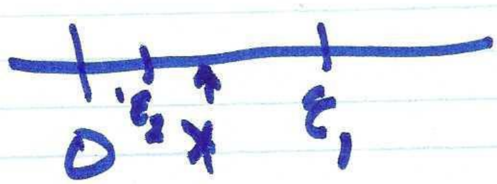
$\Rightarrow a_n \in B(B,\varepsilon)$ $|A-B| \le |A-a_n| + |B-a_n|$

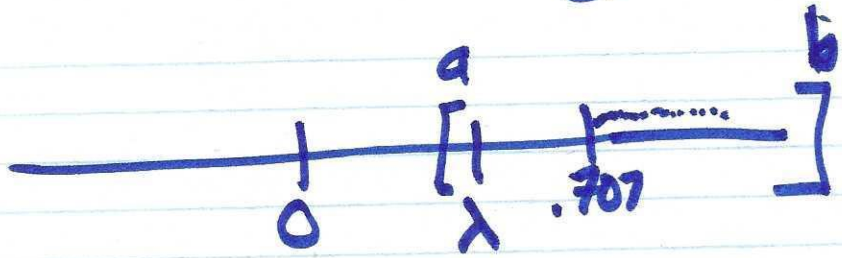
Write $d(A,B) \le d(A,a_n) + d(a_n,B)$



$\to d(A,B) \le \varepsilon + \varepsilon = 2\varepsilon$

Lemma if $0 \le x < \varepsilon$ $\forall \varepsilon > 0$, claim $x = 0$

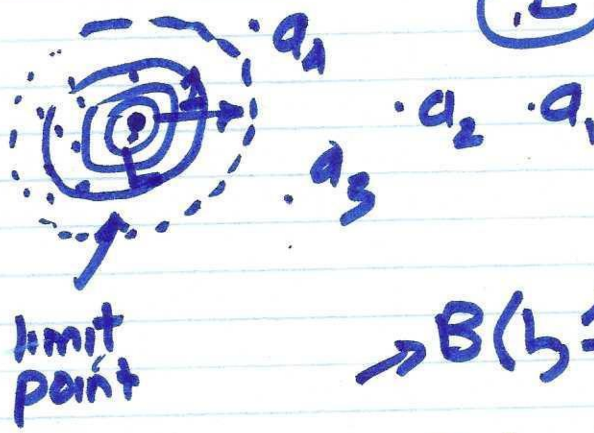




r_n

$$\left(\frac{\sqrt{2}}{2}+\frac{1}{n}\right)$$

E



Choose

$\rightarrow B(L, 1)$ a_{k_1}

$\rightarrow B(L, \frac{1}{2})$ $a_{k_2} \neq a_{k_1}$

$\rightarrow B(L, \frac{1}{4})$ $a_{k_3} \neq a_{k_1}, a_{k_2}$

TFAE

1
2
3
4

1 ⇒ 2	3
2 ⇒ 3	3
3 ⇒ 4	3
4 ⇒ 1	3
	12

A sequence is Cauchy iff

Given $\langle a_i \rangle$ & preassigned $\varepsilon > 0$,

$\exists N(\varepsilon)$ such that whenever $n, m > N(\varepsilon)$

$|a_n - a_m| < \varepsilon$.

Alternate: $|a_n - a_{n+p}| < \varepsilon \quad \forall p \in \mathbb{N}$

Big Thm: If $\langle a_i \rangle \subset \mathbb{R}$ is Cauchy, then $\langle a_i \rangle$ converges in $\mathbb{R}$.

<u>and</u> every convergent sequence is Cauchy.

Given $\varepsilon > 0$ $\exists N(\varepsilon) \ni n > N(\varepsilon) \Rightarrow$

$|a_n - L| < \varepsilon$.

$|a_n - a_m| = |a_n - L + L - a_m| \leq$

$\underbrace{|a_n - L|}_{<\varepsilon} + \underbrace{|a_m - L|}_{<\varepsilon} < 2\varepsilon = \varepsilon'$