1) Given G_α open $\forall \alpha \in \Delta$, then $\bigcup_{\alpha \in \Delta} G_\alpha$ is open
– this is an axiom for the topology

2) G_α^c is closed

Consider $\left(\bigcup_\alpha G_\alpha\right)^c$ is closed

Apply DeMorgan $\bigcap_\alpha (G_\alpha)^c$ so
any arbitrary intersection of closed sets is closed.

Note: If G_α is open $\bigcap_{\alpha=1}^{\infty} G_\alpha$ is a G-delta set

gebiet G
fermé F

If F_α is closed $\bigcup_{\alpha=1}^{\infty} F_\alpha$ is an F-sigma set

Given set E, find E' (derived set) & form $\bar{E} = \mathrm{cl}\,E = E \cup E'$

Alternatively $\bar{E} = \bigcap_{F \in \mathcal{F}} \{F \supset E, F \text{ closed}\}$

Given $\langle X, \tau \rangle$ and $Y \subset X$

the induced (relative topology) consists of each open set (in τ) intersected with Y.

So $\langle \mathbb{R}^1, \mathcal{U} \rangle$

$Y := [0, \infty)$

$\underline{[0,1)}$ $(1,2)$ $[0,1]$ closed if Y is $[0,1]$

$(\frac{1}{2}, 1]$

Today's Compactness ~ 1900 called bicompactness

"Compactness" in 19^{th} c. was what we call

today sequential compactness

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all & same in metric space

→ Borel-Lebesgue compactness

Sequential compactness

Countable compactness

Bolzano-Weierstrass compactness

Lindelöf space - any open cover reducible to ~~c~~countable subcover

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Th^{m} Compact subsets of Hausdorff spaces are closed

Hausdorff space (aka T2) has property that any two points may separated by disjoint open sets.

<u>Trennungsaxiomen</u>

T_0 .

T_1 .

increasing ↓ separation

T_2 (*)

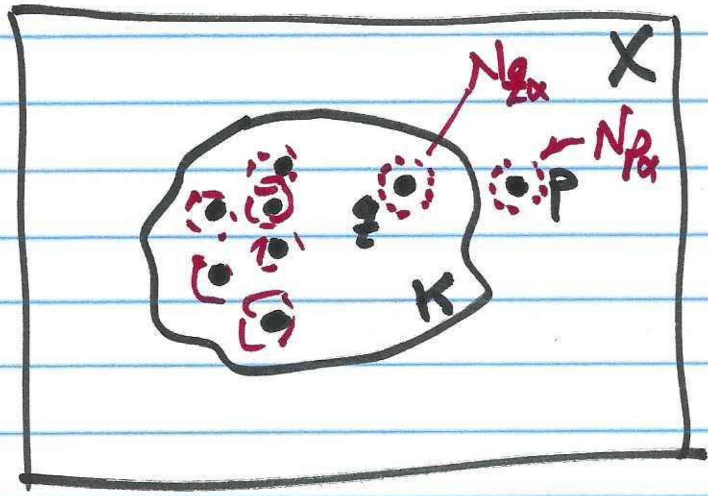
T_3 .

$T_{3\frac{1}{2}}$ ↓

T_4

T_5

Th^m 2.34 (E)



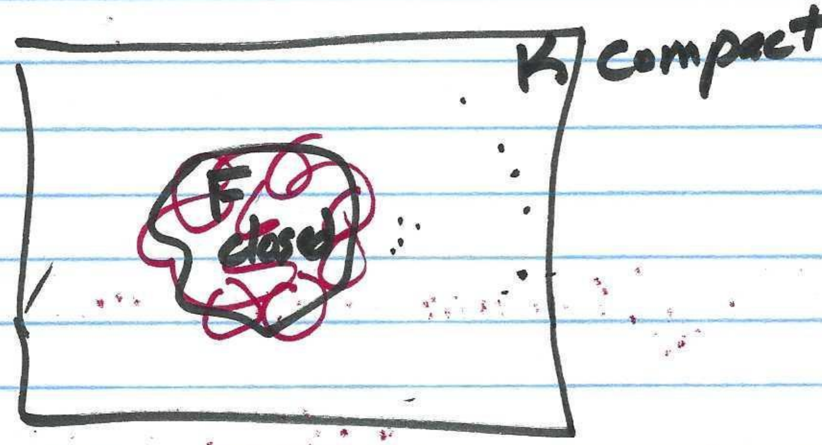
Must have finitely many N_{q_α} nbhd to cover K. Then

$$V = \bigcap_\alpha N_{p_\alpha} \text{ is still open}$$

So... $p \in V \quad V \cap K = \phi$

So $p \in \text{int } K^c$ so since p was arbitrary, K^c is open $\Rightarrow$ K is closed ∎

Thm/ Closed subset of compact set is itself compact.



Note F^c open

Suppose $\{U_\alpha\}_{\alpha \in \Delta}$ is open cover of F.

Overall cover $\{U_\alpha\} \cup F^c$ is cover of K.

Since K is compact, we may extract a subcover $\{U_1, U_2, U_3, \cdots, F^c\}$

So $\{U_1, U_2, U_3, \cdots\}$ cover F.

$\therefore$ F compact. $\blacksquare$

Heine-Borel

Lemma: Nested Interval Th^{m}

$\{[a_i, b_i]\}_{i\in N}$ where $i > j \Rightarrow [a_i, b_i] \supset [a_j, b_j]$

then, if $\lim_{i\to\infty} |b_i - a_i| = 0$,

$\bigcap_{i\in N} \{[a_i, b_i]\}$ = <u>single point.</u>

Note for every b_k, $a_i < b_k \ \forall i$

$\sup_{i\in N}\{a_i\} \le b_k$

Given closed & bounded set S

$S \subset [a, b]$

Assume FSOC that no finite subcover

of $[a,b]$ exists for given cover.

If so, consider $\underline{[a, a+\frac{(b-a)}{2}]}$ $\underline{[a+\frac{(b-a)}{2}, b]}$

One of these requires infinitely many sets in subcover.

Keep bisecting closed $(\frac{1}{2})$ intervals

Eventually, $|I_k| < \frac{1}{2^n}$ for some n

We have I_k requires infinitely many open sets to cover it. Some point α is common to all I_k's.