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1 $r \in \mathbb{Q}$ $x \notin \mathbb{Q}$

a) where is $\underline{r+x}$ Let $r = \frac{m}{n}$

$\frac{m}{n} + x = \frac{u}{v} = x = \frac{u}{v} - \frac{m}{n} = \frac{nu - mv}{vn}$

b) where $r \cdot x$

$\frac{m}{n} \cdot x = \frac{u}{v} \Rightarrow x = \frac{nu}{mv} \in \mathbb{Q} \Rightarrow\Leftarrow$

2 $\left[(\sqrt{2})^{\sqrt{2}}\right]^{\sqrt{2}}$ $(\sqrt{2})^2 \Rightarrow 2$ $a^{k^{k^{k}}}$ $2^{3^{2}}$

3 $|x| + |y| \geq |x+y|$

$|x| + |z| \geq |x+z|$ Let $z = y - x$

$|x| + |y-x| \geq |y|$

$\big||x| - |y|\big| \geq |x-y|$

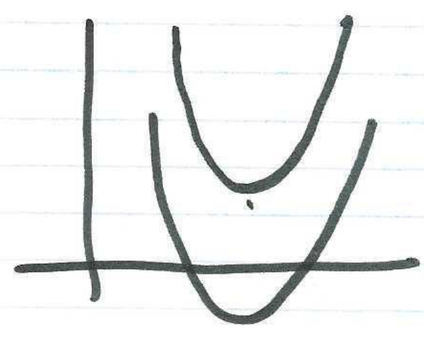
$$x = \langle x_1, x_2, x_3, x_4 \rangle \qquad y = \langle y_1, y_2, y_3, y_4 \rangle$$

$$x \cdot y = \sum_{i=1}^{4} x_i y_i \le |x||y|$$

$$-1 \le \frac{x \cdot y}{|x||y|} \le 1$$

Show $\left(\sum_{i=1}^{n} a_i b_i\right)^2 \le \sum_{i=1}^{n} a_i^2 \cdot \sum_{i=1}^{n} b_i^2$

$$\sum_{i=1}^{n} |\bar{a}_i x - b_i|^2 \ge 0$$



$$\sum_{i=1}^{n} (a_i x - b_i)^2 \ge 0$$

$$\sum_{i=1}^{n} (a_i^2 x^2 - 2a_i b_i x + b_i^2) \ge 0$$

$$\left(\sum_{i=1}^{n} a_i^2\right) x^2 - 2\left[\sum_{i=1}^{n} a_i b_i\right] x + \left(\sum_{i=1}^{n} b_i^2\right) \ge 0$$

$$\sqrt{\Delta} = \sqrt{4\left(\sum_{i=1}^{n} a_i b_i\right)^2 - 4 \sum_{i=1}^{n} a_i^2 \cdot \sum_{i=1}^{n} b_i^2}$$

$$\frac{\sqrt{\Delta}}{2} = \sqrt{\underbrace{\left(\sum_{i=1}^{n} a_i b_i\right)^2 - \sum a_i^2 \cdot \sum b_i^2}}$$

$$\Rightarrow \left(\sum_{i=1}^{n} a_i b_i\right)^2 - \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right) \leq 0$$

$$\Rightarrow \left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right)$$

$$A \cdot B \leq |A| \cdot |B|$$

Cauchy - Schwarz - Bunyakowski

Why doesn't $\mathbb{C}$ have an order like $\mathbb{R}$?

$i = \sqrt{-1}$

$i < 0 \Rightarrow -1 < 0$

$\cancel{i = 0}$

$i > 0 \Rightarrow -1 > 0$

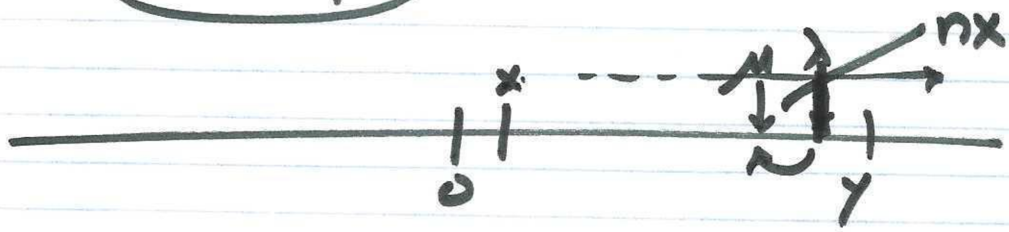
$-i > 0$

$i^2 > 0 \Rightarrow -1 > 0$

$\mathbb{R}$ has the archimedean property:

Given $x>0$ $y>0$ $\exists n \in \mathbb{N} \ni$

$nx > y$



Pf: Suppose FSOC no such n exists.

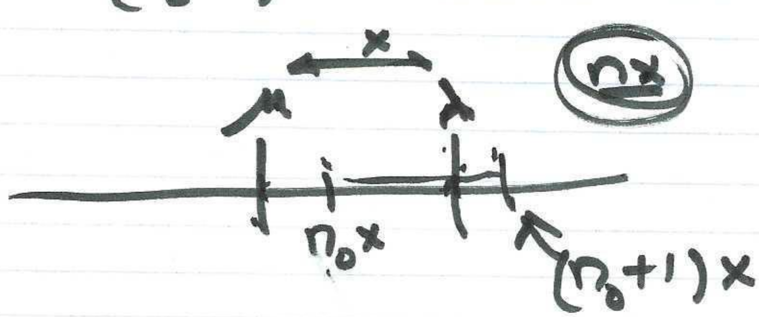
If so y is upper bound $\{nx : n \in \mathbb{N}\}$

By completeness axiom (i.e. l.u.b exists)
we have l.u.b. $\lambda > nx \ \forall n$

$\mu = (n-1)x$

Some element $n_0 x > \mu$

Consider $(n_0+1)x > \lambda$ $\rightarrow\leftarrow$



$$\mathbb{R}^n = R \times R \times \cdots R$$

$$d(x,y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$ ← euclidean "metric"

$$\langle \mathbb{R}^n, d \rangle = E^n$$ euclidean n-space

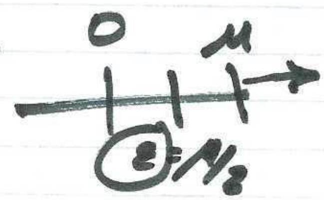
$$|x+y|^2 = (x+y)^2 = \underline{x^2 + 2xy + y^2}$$

$$2xy \le 2|x||y|$$

$$\le x^2 + 2|x||y| + y^2 =$$

$$|x|^2 + 2|x||y| + |y|^2 = (|x|+|y|)^2$$

$$|x+y|^2 \le (|x|+|y|)^2$$



If $x < \varepsilon$ for all $\varepsilon > 0$

What is x? If not, say $x > \mu > 0$

then choose $\varepsilon = \mu/2 \Rightarrow \Leftarrow$

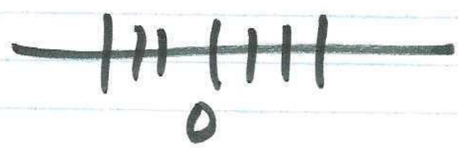
Ground space (i.e. like $\mathbb{R}$ or $\mathbb{Q}$ or $\mathbb{C}$)
Define family of sets that are "stable" under arbitrary unions and finite intersections.
So if X is ground space

$\mathcal{P}(X) = 2^X$ is set of all subsets of X.

i) on unions if $\mathcal{O}_\alpha$ belongs to family then $\bigcup_{\alpha \in \Delta} \mathcal{O}_\alpha$ also belongs to family

ii) for $\mathcal{O}_1 \cap \mathcal{O}_2 \in$ family

$\left(\frac{1}{n}, \frac{1}{n}\right)$



$\langle \{\mathcal{O}_\alpha\}_{\alpha \in \Delta}, X \rangle$ is a topology