Euler's Golden Rule:

$$c = a^{\log_a c} = b^{\log_b c}$$

$$a = b^{\log_b a}$$

$$c = \left(b^{\log_b a}\right)^{\log_a c} = b^{\log_b c}$$

$$b^{(\log_b a)(\log_a c)} = b^{\log_b c}$$

$b^x = b^y \iff x = y$ for $b > 0$ $b \neq 1$

$$\Rightarrow \log_b a \cdot \log_a c = \log_b c$$

$$\boxed{\log_a c = \frac{\log_b c}{\log_b a}}$$

Switcheroo Lemma: 2

$$\boxed{a^{\log_b c}} = c^{\log_b a}$$

$$\left(b^{\log_b a}\right)^{\log_b c} = b^{\log_b a \cdot \log_b c}$$

$\Uparrow$

a

$$= \left(b^{\log_b c}\right)^{\log_b a} = \boxed{c^{\log_b a}}$$

$$x^{\ln 4} + x^{\ln 10} = x^{\ln 25}$$

Apply Switcheroo:

$$4^{\ln x} + 10^{\ln x} = 25^{\ln x}$$

$\div 4^{\ln x}$ $\downarrow$

$$1 + \left(\frac{5}{2}\right)^{\ln x} = \left(\frac{5}{2}\right)^{2\ln x}$$

Let $u = \left(\frac{5}{2}\right)^{\ln x}$

$$1 + u = u^2$$

$$u^2 - u - 1 = 0$$

$$\rightarrow u = \frac{1 \pm \sqrt{5}}{2}$$

3

$$\left(\frac{5}{2}\right)^{\ln x} = \frac{1+\sqrt{5}}{2}$$

take logs to base 5/2

$$\ln x = \log_{\frac{5}{2}}\left(\frac{1+\sqrt{5}}{2}\right)$$

$$x = e^{\log_{5/2}\left(1+\frac{\sqrt{5}}{2}\right)} = e^{0.5252}$$

$$\sim 1.691$$

Claim: $\exists\, x \in \mathbb{R} \overset{\text{s.t.}}{\ni} x \notin \mathbb{Q}$.

Suppose $\sqrt{2} \in \mathbb{Q}$ (to the contrary)

Assume $\sqrt{2} = \frac{m}{n}$ $\quad (m,n) = 1$

$$2 = \frac{m^2}{n^2} \Rightarrow 2n^2 = m^2$$

Note: $2 \mid m$ i.e. $m = 2k$

we have $2n^2 = (\underbrace{2k}_{m})^2 = 4k^2$

$\div 2$: $n^2 = 2k^2$, now reverse argument

Since $2 \mid n^2$, $2 \mid n$ <u>but</u> $2 \mid m$ so $\Rightarrow\Leftarrow$

$\therefore \sqrt{2} \notin \mathbb{Q}$. $\blacksquare$

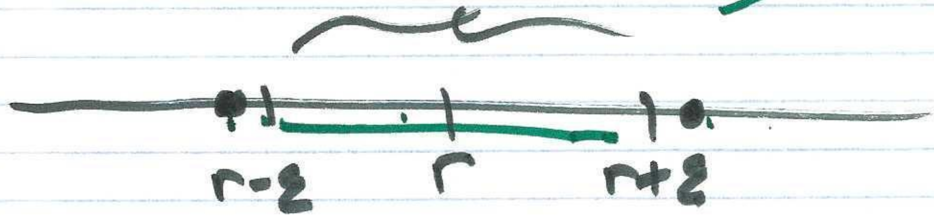
In $\mathbb{R}$, $\mathbb{Q}$ is <u>dense</u> in $\mathbb{R}$

So pick $r \in \mathbb{R}$ & tolerance $\varepsilon > 0$

Want to produce some $\alpha \in \mathbb{Q} \ni$

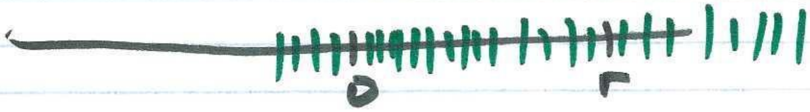
$\alpha \in (r - \varepsilon, r + \varepsilon)$

Claim that $\exists n \in \mathbb{N} \ni \frac{1}{2^n} < \varepsilon$



Consider the set $\{ q \in \mathbb{Q} \mid q = \frac{k}{2^n} \text{ for } k \in \mathbb{N} \}$

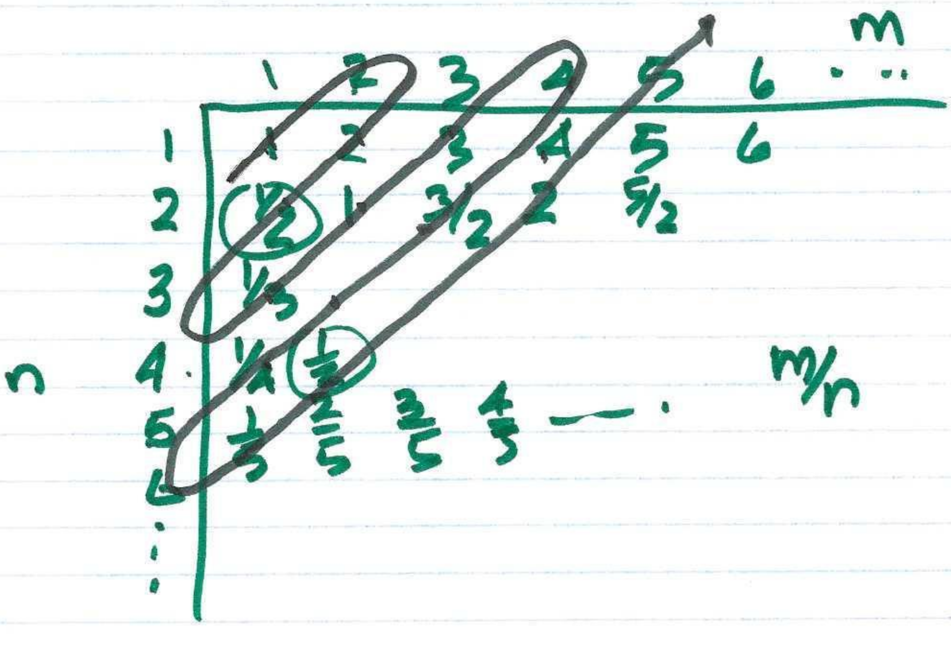
$\therefore \exists q \in \mathbb{Q} \ni q \in (r-\varepsilon, r+\varepsilon)$ by construction.



$\blacksquare$

$\mathbb{N} = \{1, 2, 3, \cdots\}$

$\mathbb{N}_0 = \{0, 1, 2, 3, \cdots\}$



6 $|A|$ #A card A   octothorp

Lemma: Show card $\mathbb{Q} \leq$ card $\mathbb{N}$

Given $\frac{m}{n} \in \mathbb{Q}$, consider $\alpha = 2^n 3^m$

Certainly $\alpha \in \mathbb{N}$

$$f : \mathbb{Q} \overset{\text{injective}}{\longmapsto} \mathbb{N} \Rightarrow \text{card } \mathbb{Q} \leq \text{card } \mathbb{N}$$

Assume card $\mathbb{R}$ = card $\mathbb{N}$

$$\begin{array}{lllllll}
1 & d_1 & d_2 & d_3 & \cdots & d_n & \cdots \\
2 & e_1 & e_2 & e_3 & \cdots & e_n & \cdots \\
3 & f_1 & \cdots & & & & \\
4 & & & & & & \\
\vdots & & & & & & \\
\infty & & & & & &
\end{array}$$

Pick $x_1 \neq d_1$ $x_2 \neq e_2$ $x_3 \neq f_3$

This construction shows card $\mathbb{R}$ > card $\mathbb{N}$