Its reach extends beyond functional analysis to:

- Proof of the existence of Green's functions [Garabedian and Schiffman 1954]
- Banach's solution of the 'easy' problem of measure [Bachman and Narici 1966, p. 188f]
- Applications to control theory [Leigh 1980, Rolewicz 1987]
- Applications to convex programming [Balakrishnan 1981]
- Applications to game theory [König 1982]
- A formulation of thermodynamics [Feinberg and Lavine 1983]

3 A short history of analysis

In the nineteenth century, 'vector' meant 'n-tuple.' Toward the end of the century, its scope was extended to include 'sequence'—for some, anyway. There were only fleeting contacts between geometric ideas and analysis for the most part and notions of proof were quite relaxed, to say the least. The geometric theorem-proof style, common today in most areas of mathematics, had to wait for the insights of Peano and Hilbert & Co. To 'prove' something, you merely stated your case and argued its plausibility. It was unfortunately similar to the rash manner in which the social 'sciences' provide 'proofs' in the modern era. We briefly illustrate how cavalier even such greats as Fourier and Euler were in this regard in Sec. 3.3.

In the period 1890-1915 notions of structure were emerging in analysis and geometric perspectives were being adopted. Standards of rigor were greatly improved and new integrals made it possible to unify several different things.

3.1 Structure

Mathematics had matured to the point where the similarities between manipulating different concrete objects were becoming apparent. A way was needed to be able to express this indifference to actual identity. The ultimate framework was to let the objects be points of an arbitrary set whose interactions were governed by a set of rules. It happened first in algebra. There, Peano [1888] defined vector space and linear map axiomatically. No more were vectors n-tuples or sequences; now you could not know exactly what the 'vectors' were. Significantly, this opened the way to vector spaces of arbitrary dimension, in particular to function spaces. But even though Pincherle wrote a book about linear spaces in 1901, Peano's idea was mostly ignored. Still, the idea of defining a space abstractly as 'objects' that obeyed certain rules was one whose time had come. *Groups* (a term coined by Galois) were defined on an arbitrary set for the first time by Weber in 1895; *field* in 1903.

In analysis it took a little longer than it did in algebra for the idea of structure to take hold. The concrete objects here were functions but confusion persisted about exactly what a function was. Dirichlet (1837) defined a numerical-valued function of a real variable to be a table, or correspondence or correlation between two sets of numbers. Riemann (1854) saw problems with the intuitive notion of function. To make the point that our understanding was too primitive, he invented a function—defined by a trigonometric series—which is continuous for irrational values of the independent variable, discontinuous for rational values. Weierstrass's (1874) classic example of a nowhere differentiable, continuous function made the point even more dramatically. As a result of these discoveries, Dedekind, Weierstrass, Méray and Cantor, by different routes, made the $\epsilon - \delta$ technique part of the standard *répertoire* of analysis.

Pincherle insisted on distinguishing between the function and the values it assumed. He said that mathematicians should use f rather than $f(x)$, to think of the function itself as an entity, divorced from its values. He and others decried the confusion between a linear map and the matrix which represented it in a particular coordinate system, a problem that is unfortunately still with us. Concomitant with the point of view that functions were entities in themselves, Volterra [1888] suggested that we should be thinking of functions defined on new domains such as on all continuous curves in a square, and doing analysis on them—no easy trick without general topology at one's disposal. He called these new kinds of functions *fonctions de ligne*, the *ligne* being the continuous curve within the square.

But what is a curve? protested Peano. The term meant something like a continuous image of $[0, 1]$ in the unit square. Peano's space-filling curve eloquently demonstrated the diverse possibilities that such a definition permitted. Hadamard was intrigued by Volterra's suggestion, however, and persisted. In 1903 he called the new functions of functions *functionals*, analysis on them *functional analysis*. Part of this was not new. In the early 1800s there was also consideration of functions whose domains were functions—derivatives, Laplace transforms, shift operators—but the radical thing at that time was applying algebraic rules to them, a notion heretofore thought only to apply to numbers. The time had now come to consider the analytic properties of such operators.

Fréchet [1904] propounded ideas of limit and continuity in sets which did not consist of numbers. In his 1906 thesis he defined the present notion of metric (He did not coin the term metric *space*, incidentally. Hausdorff introduced the more geometric-sounding nomenclature in 1913.) and investigated concrete metric spaces in which the 'points' were functions. He saw and stressed the importance of completeness, compactness and separability.

3.2 Point of view—Geometric perspective

Geometry had been 'algebraized' in the early seventeenth century by Descartes and Fermat. It was time for geometry's revenge in the late nineteenth and early twentieth, time for it to 'geometrize' analysis. Schmidt [1908] and Fréchet [1908] introduced the language of geometry into the Hilbert space ℓ_2, first spoke of the norm (in its present notation $\|x\|$) and of the triangle inequality for the norm. In 1913 Riesz described the solution of systems of homogeneous equations

$$f_i(x) = a_{i1}x_1 + \cdots + a_{in}x_n = 0,\ 1 \le i \le n$$

as an attempt to find $x = (x_1, \ldots, x_n)$ *orthogonal to* the linear span $[f_1, \ldots, f_n]$ where $f_i = (a_{i1}, \ldots, a_{in})$, i.e., he viewed solving the equations as an attempt to identify the orthogonal complement of the linear span $[f_1, \ldots, f_n]$ of the $f_1, \ldots, f_n$. Significantly, the 'equations,' the f_i, achieved vector status and stood on equal footing with the 'variables.' Hilbert and his school also spoke of *orthogonal expansions.* Helly and others, relying on earlier work of Minkowski [1896] introduced ideas about *convexity* into the blood stream of analysis. The legacy of those ideas is still very much with us.

3.3 Precision

Two principal defects of analysis in the seventeenth century were its capricious intuitiveness and its purely formal manipulation of symbols. As an example of this intuitiveness, consider Johann Bernoulli's (1693) mystic dogma that 'a quantity which is increased or decreased by an infinitely small quantity is neither increased nor decreased.' As Bishop Berkeley furiously pointed out in *The Analyst* in 1734, this gave analysts the best of both worlds: they could treat this schizophrenic 'ghost of a departed quantity' as *something* until the last step of an argument and then jettison it as *nothing.* Nowadays, some applied mathematicians retain 'the little zero' dx but discard 'higher order' terms dx^2, dx^3, etc., at moments apparently determined more by convenience than rigor.

For pure manipulation of symbols in series and products without regard to convergence,.the master was Euler. Consider his 'proof' that $e^x = \sum_{n\geq 0} x^n/n!$ by means of taking the 'limit' as $n \to \infty$ in the binomial expansion

$$\left(1+\frac{x}{n}\right)^n = 1 + x + \frac{n(n-1)}{2!}\frac{x^2}{n^2} + \frac{n(n-1)(n-2)}{3!}\frac{x^3}{n^3} + \cdots$$

This apparently did not perturb his mathematical conscience. Despite Lagrange's protests, Fourier was equally uninhibited in his 1822 classic on heat, *La theorie analytique de la chaleur.* Having developed an expansion of a certain function in a series of sines and cosines, he says 'We can extend the same results to any functions, even to those which are discontinuous and entirely arbitrary.' He formally manipulates symbols, leaving convergence to take care of itself, and obtains an expansion of an 'arbitrary' odd function in a sine series.

Though the influence of the work of Cauchy, Riemann and Weierstrass had already raised standards, the work of Hilbert and his school on the foundations of geometry elevated the standards of rigor so much that most earlier mathematical work looks shabby by comparison.

3.4 New tools: The new integrals

Considerable effort was expended in the 19th century on the problem of solution of systems of infinitely many equations in infinitely many unknowns. (Try and get mathematicians *not* to try to solve equations!) In the linear case the simultaneous linear equation problem could be stated: Given linear functionals f_i and scalars c_i, find x such that $f_i(x) = c_i$. However many f's (and c's) there were, that was the number of coordinates x was supposed to have. When there are infinitely many f's and c's, x must have infinitely many components or coordinates—must be a sequence, that is, rather than a tuple. Considerable progress in solving infinite systems of linear equations was achieved by cleverly generalizing determinants. The basic idea was to truncate the infinite system of linear equations and then take a limit. A serious weakness of the approach was its dependence on infinite products which converge only under highly restrictive circumstances. Lebesgue and Stieltjes' new theories of the integral made it possible to unify the problems, of which the following are two special cases.

1. *Fourier series.* Given a sequence (g_n) of cosines, say, and (a_n) of numbers, perforce from ℓ_2, find a function x for which these were the Fourier coefficients, i.e., such that $\int x(t)\, g_n(t)\, dt = a_n$ for every $n \in \mathbf{N}$. Is x unique?

2. *Moment problems.* Given a sequence (a_n) of numbers, find a function x such that $\int t^n x(t)\, dt = a_n$ for every $n \in \mathbf{N}$.

4 What Riesz did

Borrowing some things already done in Hilbert space, Riesz [1910, 1911] set out to solve the following problem: For $p > 1$ (so he could use the Hölder and Minkowski inequalities which he had just generalized),

(P) Given infinitely many y_s in $L_q[a, b]$ and scalars c_s,find x in $L_p[a, b]$ such that

$$\int_a^b x(t) y_s(t)\, dt = c_s$$