11/7

Counterexample:

If $f'(x_0) > 0$, does there exist a nbhd $(x_0 - \varepsilon, x_0 + \varepsilon)$ for some $\varepsilon > 0$ where f is increasing?

No.

Consider $f(x) = \begin{cases} x + x^2 \sin\left(\frac{1}{x^2}\right) & x \neq 0 \\ 0 \text{ if } x = 0 \end{cases}$

Away from 0, $f'(x) =$ $\qquad x^{-2} \to \underline{-2x^{-3}}$

$$1 + 2x\sin\left(\frac{1}{x^2}\right) - 2x^2\cos\left(\frac{1}{x^2}\right)\cdot\left(\frac{1}{x^3}\right)$$

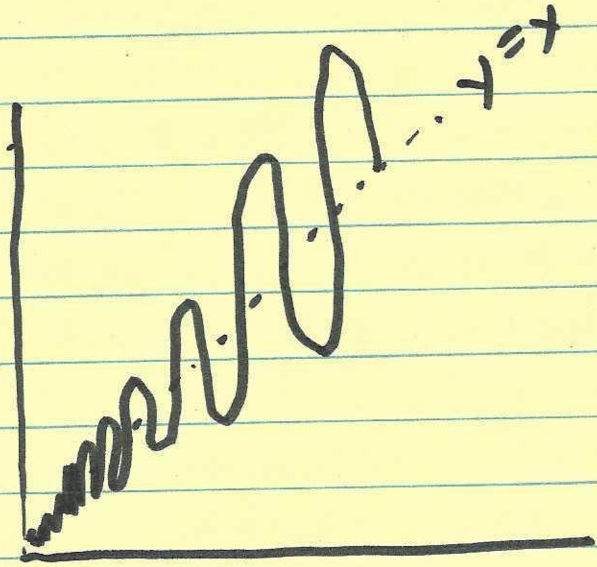
$$f'(x) = 1 + \underset{\to 0}{2x\sin}\left(\frac{1}{x^2}\right) - 2\left(\frac{1}{x}\right)\cos\left(\frac{1}{x^2}\right)$$

$$f'(0) = \lim_{h \to 0}\left(\frac{h + h^2\sin\left(\frac{1}{h^2}\right)}{h}\right) =$$

②

$$f'(0) = 1 + \lim_{h \to 0} \left(h \underbrace{\sin\left(\frac{1}{h^2}\right)}_{\to 0} \right)$$

$$f'(0) = 1$$



Oscillation:

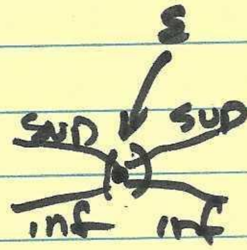
for sequence: $\langle a_n \rangle_{n \in \mathbb{N}}$

$$\omega(a_n) := \limsup_n a_n - \liminf_n a_n$$

for real function: $f(x)$ on interval I

$$\omega_f(I) = \sup_{x \in I} f(x) - \inf_{x \in I} f(x)$$

$(x_0 - \varepsilon, x_0 + \varepsilon)$



oscillation @ point $\omega_f(x_0) =$

$$\lim_{\varepsilon \to 0} \omega_f(x_0 - \varepsilon, x_0 + \varepsilon)$$

For metric space to $\mathbb{R}$ $\quad f: M \to \mathbb{R}$

E is open set in M

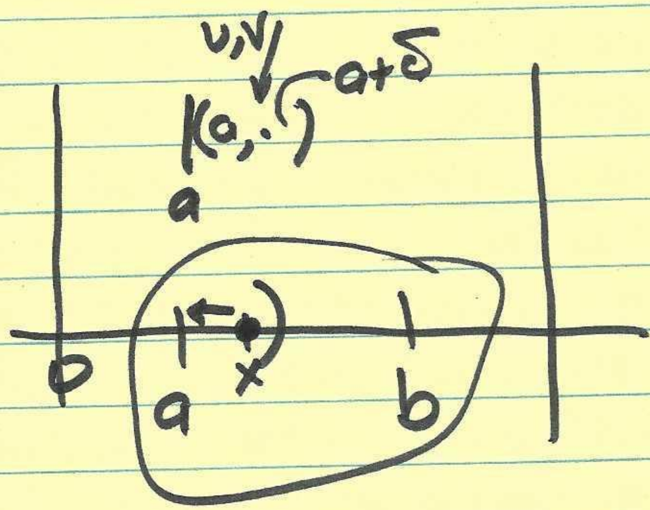
$$\omega_f(E) = \sup_{x \in E} f(x) - \inf_{x \in E} f(x)$$

@ point in M

$$\omega_f(x_0) = \lim_{\varepsilon \to 0} \omega_f(B(x_0, \varepsilon))$$

If $f(x) \to 0$ as $x \to a$ and
$g(x) \to 0$ " " "

then $\lim_{x \to a} \left(\frac{f(x)}{g(x)} \right) = \lim_{x \to a} \left(\frac{f'(x)}{g'(x)} \right) = \lambda$ if $g'(x) \neq 0$



$$\lim_{x \to a^+} f(x) = 0 \; ; \; \lim_{x \to a^+} g(x) = 0$$

we know $\lim_{x \to a^+} \left(\frac{f'(x)}{g'(x)} \right) = \lambda$ ✓

Given $\varepsilon > 0$, $\exists \, \delta(\varepsilon)$ such that

$$\cancel{\frac{f'(\xi)}{g'(\xi)}} \quad \left| \frac{f'(\xi)}{g'(\xi)} - \lambda \right| < \varepsilon \text{ if}$$

$$a < \xi < b$$

Use CMVT

Pick $u, v \in (a, a+\delta)$

$$\left|\frac{f(u)-f(v)}{g(u)-g(v)} - \lambda\right| = \left|\frac{f'(\hat{\xi})}{g'(\hat{\xi})} - \lambda\right| < \varepsilon$$

Now let v approach a thru + values

So $f(v) \to 0$ & $g(v) \to 0$ so

$$\left|\frac{f(u)}{g(u)} - \lambda\right| = \left|\frac{f'(\hat{\xi})}{g'(\hat{\xi})} - \lambda\right| < \varepsilon$$

$$\lim_{u\to a^+} \frac{f(u)}{g(u)} = \frac{f(a)}{g(a)} = \cancel{\frac{f'(\xi)}{g'(\xi)}} - \lim_{\xi\to a^+}\frac{f'(\xi)}{g'(\xi)} = ?$$

$$\frac{f'(a)}{g'(a)}$$

What is $\lim_{x \to 0} \left(1 + x\right)^{\frac{1}{x}} \leftarrow f(x) \quad \sim 1^{\infty}$

Take ln: $\lim_{x \to 0} \ln f(x) = \frac{1}{x} \ln(1+x)$

? $\lim_{x \to 0} \frac{\ln(1+x)}{x} = \lim_{x \to 0} \left(\frac{1/1+x}{1} \right) = 1$

$$\ln f(x) = 1 \Rightarrow f(x) = e^{x}$$

So we recover $e^{x} = \lim_{x \to 0} \left(1+x\right)^{\frac{1}{x}}$

$\lim_{x \to 0} \frac{\sin x}{x} = 1$ ✓

$\lim_{x \to 0} \cos x = \cos \text{"}0\text{"} = 1$

$\lim_{x \to 0} \frac{\cos x}{1} = \frac{1}{1} = 1$

$(\sin x)'\Big|_{a} = \lim_{\Delta x \to 0} \left(\frac{\sin(a + \Delta x) - \sin a}{\Delta x} \right)$

$$\lim_{\Delta x \to 0} \left(\frac{\sin a \cos \Delta x + \cos a \sin \Delta x - \sin a}{\Delta x} \right)$$

$$\lim_{\Delta x \to 0} \left(\frac{\sin a \left(\cos \Delta x - 1\right) + \cos a \sin \Delta x}{\Delta x} \right)$$

$$\frac{\sin \Delta x}{\Delta x} = \frac{\Delta x - \frac{(\Delta x)^3}{3!} + O(\Delta x)^5}{\Delta x}$$

$$= 1 - \frac{\Delta x^2}{6} + O(\Delta x)^4$$