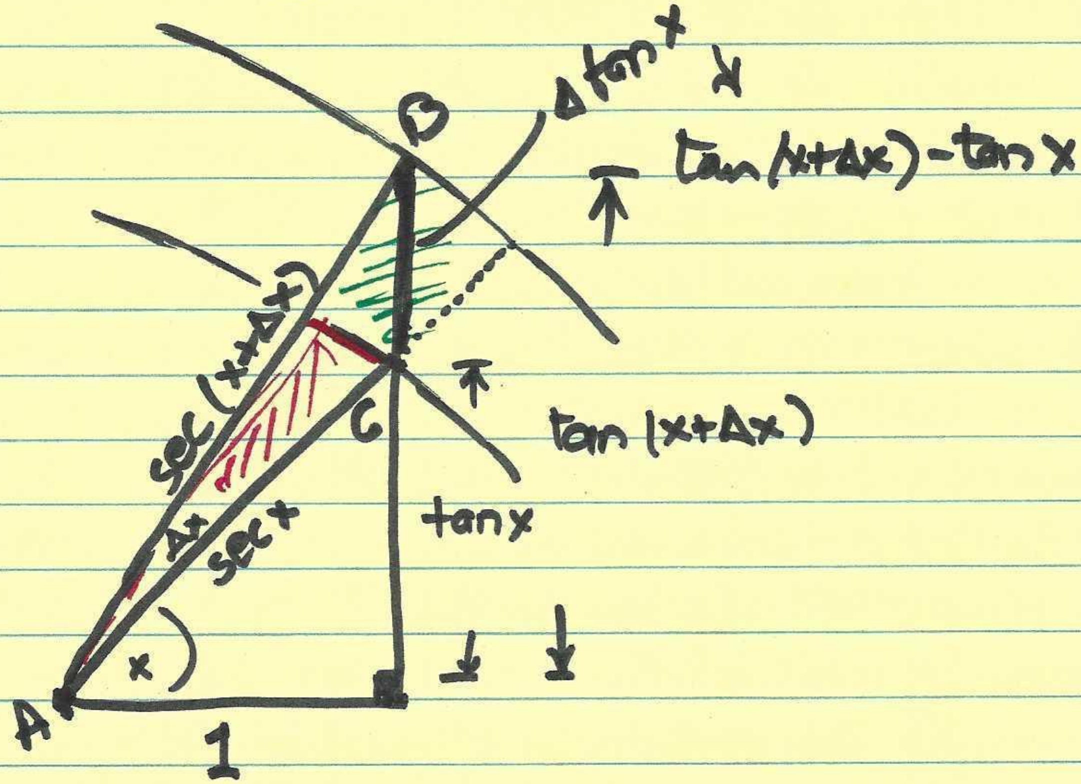
11/12



Area in Red   Area ΔABC   Area Red+Green

$$\frac{1}{2}\Delta x \cdot \sec^2 x \le \frac{1}{2}\,\frac{\Delta \tan x}{2} \le \frac{1}{2}\Delta x \cdot \sec^2(x+\Delta x)$$

$$\sec^2 x \le \frac{\Delta \tan x}{\Delta x} \le \sec^2(x+\Delta x)$$

Let $\Delta x \to 0$

$$\underline{\sec^2 x} \le \lim_{\Delta x \to 0}\left(\frac{\Delta \tan x}{\Delta x}\right) \le \underline{\sec^2 x}$$

②

Recall $\tan^2 x = \sec^2 x - 1$

$$2\tan x(\tan x)' = \left[(\cos x)^{-2}\right]'$$

$$\not{2}\tan x(\sec^2 x) = -\not{2}(\cos x)^{-3}(\cos x)'$$

$$\frac{-\sin x}{\cos^3 x} = \frac{(\cos x)'}{\cos^3 x}$$

$\Rightarrow$ $\frac{d}{dx}(\cos x) = -\sin x$

$$(\tan x)' = \left(\frac{\sin x}{\cos x}\right)'$$

$$\frac{1}{(\cos x)^2} = \frac{\cos x(\sin x)' - \sin x(\cos x)'}{(\cos x)^2}$$

$$1 = \cos x(\sin x)' + \sin^2 x$$

$$1 - \sin^2 x = \cos x(\sin x)'$$

$$\cos x = (\sin x)'$$

Matrix Derivatives (vector-valued functions)

Frechet Def'n

$f : \mathbb{R}^n \to \mathbb{R}^m$

Let $x_0 \in \underset{\sim}{U} \in \mathbb{R}^n$

We say f is differentiable @ x_0 if $\exists\, f'(x_0)$

[which is a linear map] from $\mathbb{R}^n \to \mathbb{R}^m$

and a function $r : U \to \mathbb{R}^m$

where r is continuous @ x_0 and $r(x_0) = 0$

$\ni$ $f(x) = f(x_0) + \underline{f'(x_0)}(\underline{x - x_0}) + r(x)\|x - x_0\|$

Caratheodory

$f(x)$ differentiable @ x_0 iff $\exists\, \varphi(x)$,

which is matrix-valued, depending on

x_0 & is continuous @ x_0 such that

$$f(x) = f(x_0) + \varphi(x)(x - x_0)$$

then $f'(x_0) = \varphi(x_0)$

$f : \mathbb{R}^n \to \mathbb{R}$

$$df = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right] \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix}$$

$\langle \cdot | \cdot \rangle : V \times V \to \mathbb{R}$

(1) Linear in first argument

$$\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$$

(2) (Conjugate <u>symmetric</u>)

$$\langle x, y \rangle = \langle y, x \rangle$$

(3) Positive definite, if $x \neq 0$, $\langle x, x \rangle > 0$

Consider $f: \mathbb{R}^2 \to \mathbb{R}$

Can f have partials in all directions except @ origin.

$$f(x_1, x_2) = \begin{cases} \dfrac{x_1 x_2}{x_1^2 + x_2^2}, & \text{if } x_1^2 + x_2^2 > 0 \\ 0 & \text{else} \end{cases}$$

$$\left.\frac{\partial f}{\partial x_1}\right|_{(x_1, x_2) = (0,0)} = \lim_{h \to 0} \frac{f(h) - 0}{h}$$

$$\frac{\partial f}{\partial x_1} = \lim_{h \to 0} \left(\frac{f(x_1 + h, x_2) - f(x_1, x_2)}{h} \right)$$

Let $x_2 = m x_1$ $\quad \dfrac{(x_1)(m x_1)}{x_1^2 + (m x_1)^2} = \dfrac{m \cancel{x_1^2}}{\cancel{x_1^2}(1+m)}$