Differentiation

1) Cauchy Definition (1821)

Let $I \subset \mathbb{R}$ be an interval and $x_0 \in I$. The function $f : I \to \mathbb{R}$ is differentiable at x_0 if the limit $\lambda = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$ exists. The value of this limit is the derivative of f at x_0 and is denoted by $f'(x_0)$.

2) Weierstraß Definition (1861)

Let $I \subset \mathbb{R}$ be an interval and $x_0 \in I$. The function $f : I \to \mathbb{R}$ is differentiable at x_0 **if and only** if there exists a number λ and a function $r(x)$, continuous at x_0 and satisfying $r(x_0) = 0$, such that $f(x) = f(x_0) + \lambda(x - x_0) + r(x)(x - x_0)$. The value of this number is the derivative of f at x_0 and is denoted by $f'(x_0)$.

3) Carathéodory Definition (1950)

Let $I \subset \mathbb{R}$ be an interval and $x_0 \in I$. The function $f : I \to \mathbb{R}$ is differentiable at x_0 **if and only** if there exists a function $\varphi(x)$, continuous at x_0, such that $f(x) = f(x_0) + \varphi(x)(x - x_0)$. In this case, $\varphi(x_0)$ is the derivative of f at x_0 and is denoted by $f'(x_0)$.

Comments:

(i) Cauchy's definition is, of course, the standard one used in calculus texts.

(ii) The usefulness of Weierstraß' definition is that it does not reference a limit directly but relies on the continuity of an auxiliary function $r(x)$. It is the basis for extending the notion of differentiability to functions of several variables.

(iii) Carathéodory's definition seems like a trivial adjustment to Weierstraß' definition, since all that really changes is $\varphi(x) = \lambda + r(x)$, but it is beneficial theoretically, as it simplifies proving some important results. It is immediate that if f is differentiable at x_0 then it is continuous there. Note that $\varphi(x) = \frac{f(x) - f(x_0)}{x - x_0}$ for $x \neq x_0$ is uniquely determined and if $f'(x_0)$ exists, it is also uniquely determined. $\varphi(x)$ is nothing more than the difference quotient for f set up at the point x_0, so we are saying that $f'(x_0)$ exists if its difference quotient is continuous at x_0.

Theorem: Cauchy implies Weierstraß

Proof: Given that $f : I \to \mathbb{R}$ is differentiable at x_0, then $\lambda = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$ exists. So define $r(x) = \frac{f(x) - f(x_0)}{x - x_0} - \lambda$, and then $\lim_{x \to x_0} r(x) = 0$. Multiplying by $x - x_0$, we get $r(x)(x - x_0) = f(x) - f(x_0) - \lambda(x - x_0)$. Then $f(x) = f(x_0) + \lambda(x - x_0) + r(x)(x - x_0)$. This means if

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$$f(x) = \lim_{n\to\infty} \lim_{m\to\infty} \left(\cos(2\pi n! x)\right)^{2m} \; ; \; x \in I$$

Look @ $f(x) = \begin{cases} x\sin\frac{1}{x} & x \neq 0 \\ 0 & \text{else} \end{cases}$

$x \neq 0$ $f'(x) = \sin\frac{1}{x} + x\cos\frac{1}{x}\left(-\frac{1}{x^2}\right)$

$$= \sin\frac{1}{x} - \frac{\cos\frac{1}{x}}{x}$$

$x = 0$ $\lim_{h\to 0}\left(\frac{(x+h)\sin\frac{1}{(x+h)} - 0}{h}\right)$

$$\lim_{h\to 0}\left(\frac{h\sin\frac{1}{h}}{h}\right) = \lim_{h\to 0} \sin\frac{1}{h}$$

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$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & x \neq 0 \\ 0 \quad \text{if} & x = 0 \end{cases}$$

$\underline{x \neq 0}$ $\quad f'(x) = 2x \sin \frac{1}{x} - x^2 \cos\left(\frac{1}{x}\right) \cdot \frac{1}{x^2}$

$$\rightarrow = 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)$$

$x = 0 \qquad \lim\limits_{h \to 0} \left(\frac{(0+h)^2 \sin\left(\frac{1}{h}\right)}{h} \right)$

$$\lim_{h \to 0} \left(h \underbrace{\sin\left(\frac{1}{h}\right)}_{\leq h} \right) = 0$$

Fermat's Th^m: If f is defined on $[a,b]$ and f achieves a max/min @ $x \in (a,b)$, then if f is diff on (a,b), $f'(x) = 0$.

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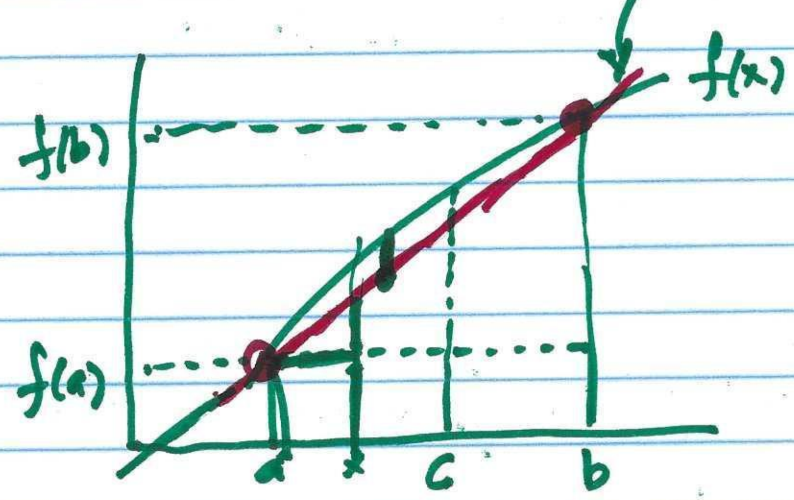

f is cont on [a,b]
diff on (a, b)

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$ slope of red line

$g(x) =$

$(a, f(a)), (b, f(b))$

$$\frac{y-f(b)}{x-f(a)} = \frac{f(b)-f(a)}{b-a}$$

$$y - f(b) = \frac{(x-f(a))}{b-a}(f(b)-f(a))$$

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$$y = \frac{f(b)-f(a)}{b-a}x + \left(f(b) - \frac{f(a)\cdot(f(b)-f(a))}{b-a}\right)$$

$$g(x) = f(a) + (x-a)\underbrace{\left(\frac{f(b)-f(a)}{b-a}\right)}_{m}$$

$$\overset{h(x)}{f(x)-g(x)} = f(x) - \left[f(a) + (x-a)\cdot\frac{f(b)-f(a)}{b-a}\right]$$

$$h(a) = f(a) - [f(a) + 0] = 0$$

$$h(b) = f(b) - f(a) - \cancel{(b-a)}\left(\frac{f(b)-f(a)}{\cancel{b-a}}\right) = 0$$

By Rolle $\exists c \in (a,b) \ni$.

$$h'(c) = 0$$

$$0 = h'(c) = f'(c) - \frac{f(b)-f(a)}{b-a}$$

So $f'(c) = \dfrac{f(b)-f(a)}{b-a}$ ✓