Assume $a_i \geq 0$ all $i \in \mathbb{N}$

Comparison Test:

If $\sum_n U_n < \infty$ and for $n > N$ it is true that $U_n \geq a_n$, then $\sum_n a_n$ converges.

$0 \leq a_n \leq U_n$ $\qquad$ $0 \leq V_n \leq a_n$ If $\sum_n V_n$

Pf: Since $\sum_n U_n = L$, $\sum a_n$ is monotone & bounded above, hence by MCT the "test" series converges.

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Cauchy Root Test for $n > N$

If $\lim_{n\to\infty} \sqrt[n]{|a_n|} < 1$ then $\sum a_n < \infty$

$> 1$ then $\sum a_n = \infty$

$= 1$ no conclusion

If $\lim_{n\to\infty} a_n^{1/n} \leq r < 1$

$$a_n \leq r^n$$

$$\begin{array}{ccccccccc} a_1 & + a_2 & + a_3 & \cdots & a_n & + a_{n+1} & \cdots \\ \wedge\!| & \wedge\!| & \wedge\!| & & \wedge\!| & \wedge\!| & \\ r & + r^2 & + r^3 & \cdots & r^n & + r^{n+1} & \cdots \end{array}$$

$$\sum_{n=1}^{\infty} r^n = \frac{r}{1-r}$$

So by Comp Test $\sum_n a_n < \infty$ $=\infty$

Likewise if $a_n \geq r^n$ — divergence

D'Alembert Ratio Test

If $\forall n > N$ it is true that

$\lim_{n\to\infty} \left| \frac{U_{n+1}}{U_n} \right| < 1$ then $\sum_n U_n < \infty$

$> 1$ " $\sum_n U_n = \infty$

$= 1$ " no info

$r < 1$

$$\frac{U_2}{U_1} < r \Rightarrow \quad U_2 < rU_1$$

$$\frac{U_3}{U_2} < r \qquad U_3 = rU_2 = r(rU_1) = r^2U_1$$

$$\frac{U_4}{U_3} < r \qquad U_4 = r^3U_1$$

$$U_n = r^{n-1}U_1$$

$$\begin{array}{ccccccccc}
U_1 & + & U_2 & + & U_3 & + & \cdots & U_n & + \cdots \\
\shortparallel & & \wedge\!| & & \wedge\!| & & & \wedge\!| & \\
U_1 & + & rU_1 & + & r^2U_1 & + & \cdots & r^{n+1}U_1 & + \cdots
\end{array}$$

$$= U_1\left[\underbrace{1 + r + r^2 + \cdots r^n + r^{n+1} + \cdots}_{\frac{1}{1-r}}\right] < \infty$$

By MCT $\sum_n U_n < \infty$

$$S_1 : \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots \frac{1}{2^n} + \cdots = 1$$

$$S_2 : \frac{1}{2} \Big| + \overset{2}{\frac{1}{8}} + \overset{1/8}{\frac{1}{4}} + \overset{2}{\frac{1}{32}} + \overset{1/8}{\frac{1}{16}} + \overset{2}{\frac{1}{128}} + \overset{1/8}{\frac{1}{64}}$$

$$\overset{1}{\frac{1}{2^1}} \Big/ + \overset{2}{\frac{1}{2^3}} + \overset{3}{\frac{1}{2^2}} + \overset{4}{\frac{1}{2^5}} + \overset{5}{\frac{1}{2^4}} + \overset{6}{\frac{1}{2^7}} + \overset{7}{\frac{1}{2^6}} + \cdots$$

Apply Root Test

n odd $\quad \frac{1}{2^{n-1}} \qquad \left(\frac{1}{2}\right)^{\frac{n-1}{n}} = \frac{1}{2}^{1-\frac{1}{n}}$

n even $\quad \frac{1}{2^{n+1}} \qquad \left(\frac{1}{2}\right)^{\frac{n+1}{n}} = \frac{1}{2}^{1+\frac{1}{n}}$

Root confirms convergence $S_2$

Ratio Test indecisive.

Prob: $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2^n} \qquad U_{n+1} = \frac{n+1}{2^{n+1}}$

$$U_n = \frac{n}{2^n}$$

$$\frac{u_{n+1}}{u_n} = \frac{n+1}{2^{n+1}} \cdot \frac{2^n}{n} = \frac{1}{2}\,\frac{n+1}{n}$$

$$\lim_{n\to\infty}\left(\frac{1}{2}\,\frac{n+1}{n}\right) = \frac{1}{2} \qquad 1+\frac{1}{n}$$

Prob: $\sum_n \frac{n^2}{2^n} \qquad \frac{(n+1)^2}{2^{n+1}},\ \frac{n^2}{2^n}$

$$\frac{n^2+2n+1}{2^{n+1}} \cdot \frac{2^n}{n^2} = \frac{1}{2}\underbrace{\left(1+\frac{2}{n}+\frac{1}{n^2}\right)}_{1} = \frac{1}{2}$$

Prob: $\sum_n \frac{n^k}{2^n} \qquad \frac{(n+1)^k}{2^{n+1}},\ \frac{n^k}{2^n}$

$$\frac{u_{n+1}}{u_n} = \frac{(n+1)^k}{\cancel{2}^{n+1}} \cdot \frac{\cancel{2^n}}{n^k} = \frac{1}{2}\left[\frac{(n+1)^k}{n^k}\right]$$

$$= \frac{1}{2}\left[\frac{n+1}{n}\right]^k$$ lim is still $\frac{1}{2} < 1$

Kummer's Th$^{m}$ (Test)

$\sum_n U_n$, test series $\sum_n a_n$

Set $U_n, a_n > 0$

① $\sum_n U_n < \infty$ if $\lim_{n\to\infty}\left(a_n \frac{U_n}{U_{n+1}} - a_{n+1}\right) \geq C > 0$

② If $\sum a_n^{-1} = \infty$ and $\lim_{n\to\infty}\left( \quad '' \quad \right) \leq 0$

then $\sum_n U_n = \infty$

Typical $a_n = \frac{1}{n \ln n}$ gives Gauss' Test

$$\frac{U_n}{U_{n+1}} = \frac{n^2 + a_1 n + a_0}{n^2 + b_1 n + b_0}$$

If $a_1 - b_1 > 1$ $\quad \sum U_n < \infty$

If $a_1 - b_1 \leq 1$ $\quad \sum U_n = \infty$

$$\left(a_n \frac{u_n}{u_{n+1}} - a_{n+1}\right) \geq C$$

$$a_n u_n - a_{n+1} u_{n+1} \geq C u_{n+1}$$

$$\frac{1}{C}\left[a_n u_n - a_{n+1} u_{n+1}\right] \geq u_{n+1}$$

Starting @ $n = N$

(i) $$\frac{1}{C}\left[a_N u_N - \cancel{a_{N+1} u_{N+1}}\right] \geq u_{N+1}$$

(ii) $$\frac{1}{C}\left[\cancel{a_{N+1} u_{N+1}} - \cancel{a_{N+2} u_{N+2}}\right] \geq u_{N+2}$$

(iii) $$\frac{1}{C}\left[\cancel{a_{N+2} u_{N+2}} - \cancel{a_{N+3} u_{N+3}}\right] \geq u_{N+3}$$

$$\vdots$$

$1, 2, \cdots N, \cdots n \longrightarrow$

($\infty$) $$\frac{1}{C}\left[\cancel{a_n u_n} - \cancel{a_{n+1} u_{n+1}}\right]$$

$$\frac{1}{C}\left[a_N u_N - \cancel{a_{n+1} u_{n+1}}\right] \geq \sum_{j=N}^{n}$$

Leibniz Criterion

If $a_n \downarrow 0$ $\quad \left\{\frac{1}{n}\right\}$

Then $\sum_{n=1}^{\infty} (-1)^{n+1} a_n < \infty$

Look @ remainder after the 2n term

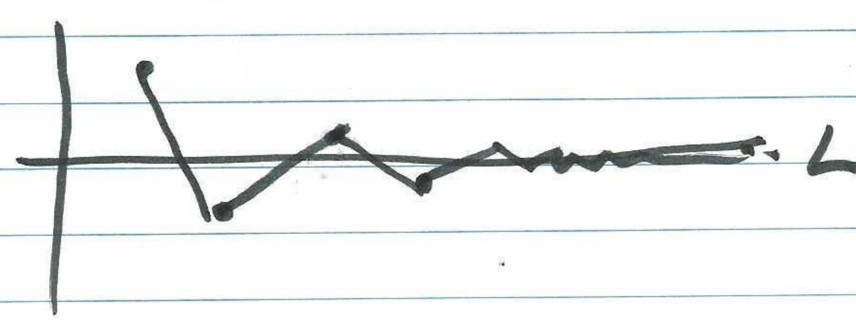
(i) $R_{2n} = (a_{2n+1} - a_{2n+2}) + (a_{2n+3} - a_{2n+4}) + \cdots$

also write as (ii) $= a_{2n+1} - (a_{2n+2} - a_{2n+3}) - (a_{2n+4} - a_{2n+5}) + \cdots$

(i) $R_{2n} > 0$

(ii) $R_{2n} < a_{2n+1}$ ← can make arbitrarily small

Conclude $R_{2n}$ is positive & bounded



$$f(x) = \ln(1+x)$$

$$\ln(x+1)$$

$$\frac{1}{x+1}$$

$$-\frac{1}{(x+1)^2}$$

$$+\frac{2}{x^3} \cdot \frac{2}{(x+1)^3}$$

$$f(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2} + \frac{f'''(0)x^3}{6} + \cdots$$

$$\ln x =$$

$$\ln x = \ln 2 + x - \frac{x^2}{2} + \frac{2}{6}(1)x^3$$

$$\ln x = \ln 2 + x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \cdot$$

$x=1$

$$0 = \ln 2 + 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} \cdots$$

$$\ln 2 = \underbrace{-1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \cdots}$$

$$-\ln 2 = \sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{1}{n}$$