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Thomae's Function

$f : [0,1] \to [0,1]$

$$f(x) = \begin{cases} 0 & \text{if } x \notin \mathbb{Q} \\ \frac{1}{q} & \text{if } x \in \mathbb{Q} \text{ ; } p/q \end{cases}$$

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a.e. almost everywhere

i.e. except on set of measure zero $\underline{\mu(S) = 0}$

Lemma $\mu(\mathbb{Q}) = 0$:

Let $r_1, r_2, r_3, \cdots, r_n, \cdots$ be an enumeration of $\mathbb{Q}$.

Construct a nbhd of width $\varepsilon > 0$ around r_1. so $(r_1 - \frac{\varepsilon}{2}, r_1 + \frac{\varepsilon}{2})$, then construct a nbhd about r_2 : $(r_2 - \frac{\varepsilon^2}{2}, r_2 + \frac{\varepsilon^2}{2})$ .. continue so r_n : $(r_n - \frac{\varepsilon^n}{2}, r_n + \frac{\varepsilon^n}{2})$ + so forth

So $\mathbb{Q} \subset \bigcup_n N_{r_n}$

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What is $\mu\left(\bigcup_{n=1}^{\infty} N_{r_n}\right)$?

Allowing for overlaps,

$$\mu\left(\bigcup_{n=1}^{\infty} N_{r_n}\right) \leq \varepsilon+\varepsilon^2+\cdots\varepsilon^n+\cdots \rightarrow$$

$$= \frac{\varepsilon}{1-\varepsilon}$$

Consider $\lim_{\varepsilon\to 0} \mu\left(\bigcup_{n=1}^{\infty} N_{r_n}\right) \leq \lim_{\varepsilon\to 0} \frac{\varepsilon}{1-\varepsilon} = \varepsilon = 0$

Pf: Consider $x \notin \mathbb{Q}$. Claim: Given $\varepsilon > 0$ $\exists\, \delta(\varepsilon) \ni f(x') < \varepsilon$, if $x' \in (x-\delta(\varepsilon), x+\delta(\varepsilon))$

We can arrange for all $\frac{1}{q} > \varepsilon$ to be excluded from $(x-\delta(\varepsilon), x+\delta(\varepsilon))$ by a finite sorting process. Then $x' \in$ nbhd $f(x') < \varepsilon$ but $f(x) = 0$, so we say $f(x)$ is continuous @ $x \notin \mathbb{Q}$. ■

<u>Heine's Th^m</u>

Continuous functions on closed & bounded intervals of $\mathbb{R}$ are uniformly continuous.

Pf:

Note: $f(x)$ is uniformly continuous on $E \subseteq \mathbb{R}$ if given $\varepsilon > 0$ $\exists\, \delta(\varepsilon) \ni |x-y| < \delta(\varepsilon) \Rightarrow$ $|f(x) - f(y)| < \varepsilon$ $\forall x, y \in E$.

Define $I_x = \left(x \pm \frac{\delta(\varepsilon, x)}{2}\right)$ $\forall x \in [a,b]$

Claim $\bigcup_x I_x$ is open cover of $[a,b]$

By Heine-Borel interval $[a,b]$ is compact, so a <u>finite</u> collection $\{I_{x_1}, I_{x_2}, I_{x_3}, \dots I_{x_n}\}$ covers $[a,b]$. Define $\delta(\varepsilon) := \mathrm{MIN}\left[\frac{\delta(\varepsilon, x_1)}{2}, \frac{\delta(\varepsilon, x_2)}{2}, \dots, \frac{\delta(\varepsilon, x_n)}{2}\right]$.

So now given x' & $x'' \in [a,b]$, we know x' is in <u>some</u> I_{x_j}

So $|x' - x_j| < \frac{\delta(\varepsilon, x_j)}{2}$.

Also, $x'' - x_j = x'' - x' + x' - x_j$, so

$$|x'' - x_j| \leq \underbrace{|x'' - x'|}_{< \delta(\varepsilon)} + \underbrace{|x' - x_j|}_{< \frac{\delta(\varepsilon, x_j)}{2}} < \delta(\varepsilon, x_j)$$

So we have $|x'' - x_j| < \delta(\varepsilon, x_j)$ so

$|f(x'') - f(x_j)| < \varepsilon$ and $|f(x') - f(x_j)| < \varepsilon$

$$\underbrace{|f(x'') - f(x')|}_{< 2\varepsilon} = \underbrace{|f(x'') - f(x_j)|}_{< \varepsilon} + \underbrace{|f(x_j) - f(x')|}_{< \varepsilon}$$

So if $|x'' - x'| < \delta(\varepsilon)$, $|f(x'') - f(x')| < 2\varepsilon$

Clean up step let $\varepsilon' = 2\varepsilon$ and $\delta'(\varepsilon) = \delta(\frac{\varepsilon}{2})$

then $|x'' - x'| < \delta'(\varepsilon) \Rightarrow |f(x'') - f(x')| < \varepsilon'$.

$\blacksquare$