10/29

Continuity in metric spaces (mostly $\mathbb{R}^k$)

$f : \langle X, d \rangle \to \langle Y, \rho \rangle$

We say f is continuous @ $x_0 \in X$ if:

(i) $\lim_{x \to x_0} f(x)$ exists

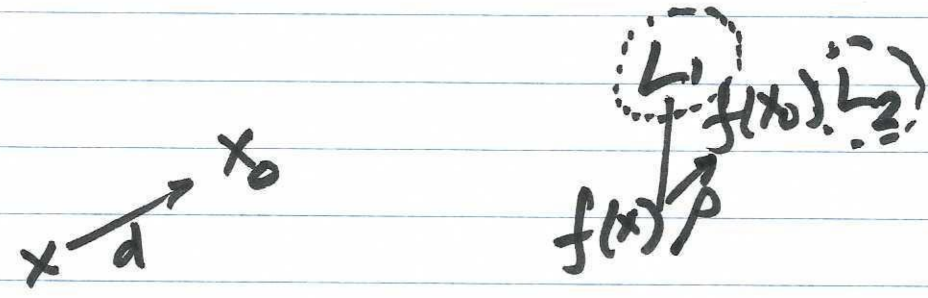
(ii) $f(x_0)$ exists

(iii) (i) = (ii)

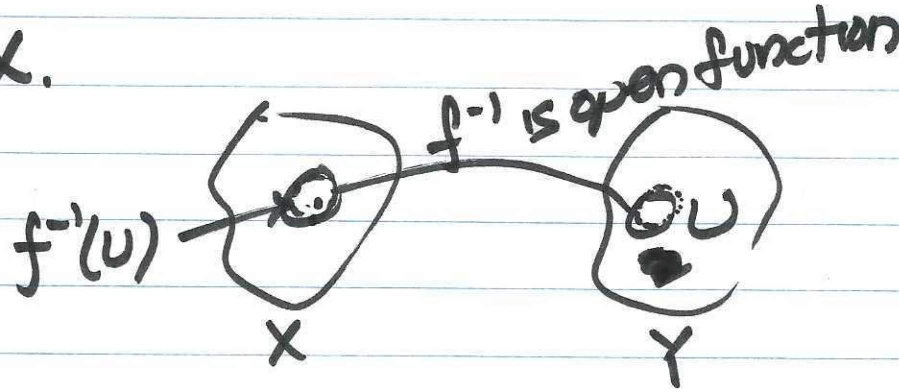
Express $\lim_{x \to x_0} f(x) = L$

Fix $\varepsilon > 0$. Need to find $\delta(\varepsilon, x_0) > 0$ such that $d(x, x_0) < \delta(\varepsilon, x_0)$ implies $\rho(f(x), L) < \varepsilon$

Lemma: If X is Hausdorff, then all limits are unique.



(global) Def^n $f : \langle X, d\rangle \to \langle Y, \rho\rangle$ is continuous iff U (open in Y) has open pre-image in X.



f

X	Y	
open	open	f is open
closed	closed	f is closed
open	← open	f^{-1} open / f cont
closed	← closed	f^{-1} closed / f cont

Commutation of Set Operations w/ Maps

$f: X \to Y \qquad A \subset X, \; B \subset Y$

$f(A \cup B) \qquad f(A) \cup f(B)$

$f(A \cap B) \subseteq f(A) \cap f(B)$

$f(A^c) \subseteq [f(A)]^c$

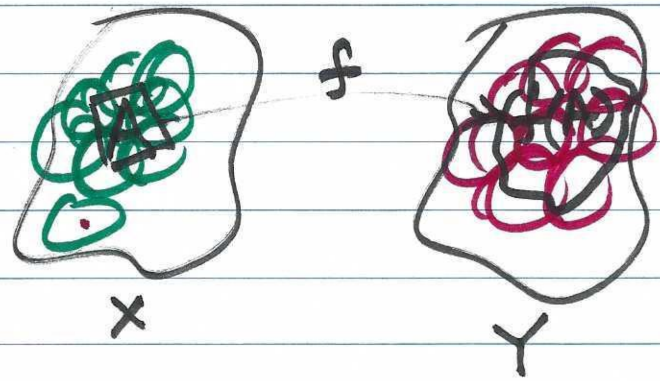
$$\text{behaved} \left\{ \begin{array}{l} f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B) \\ f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B) \\ f^{-1}(A^c) = [f^{-1}(A)]^c \end{array} \right.$$

$f^{-1}(B) \subset X$

$f^{-1}(B)$ is f-saturated

(A)

Continuous functions preserve compactness



A compact, so extract finite subfamily of open sets.

So finitely many forward images cover $f(A)$, i.e. $f(A)$ is compact

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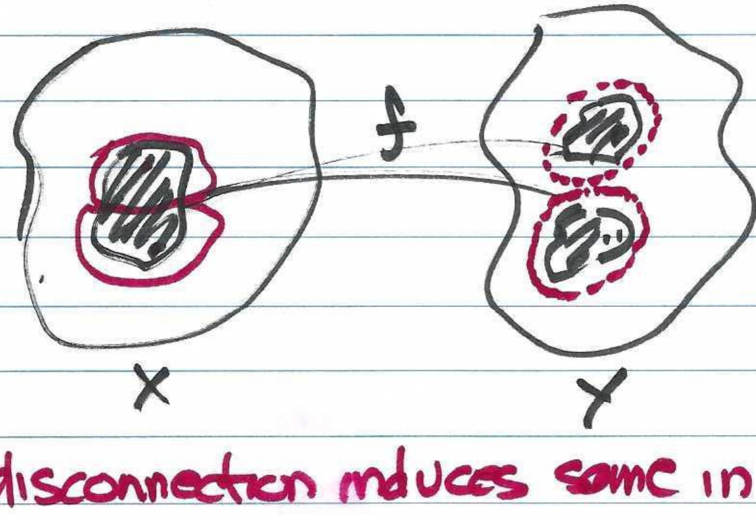
Continuous functions preserve connectedness

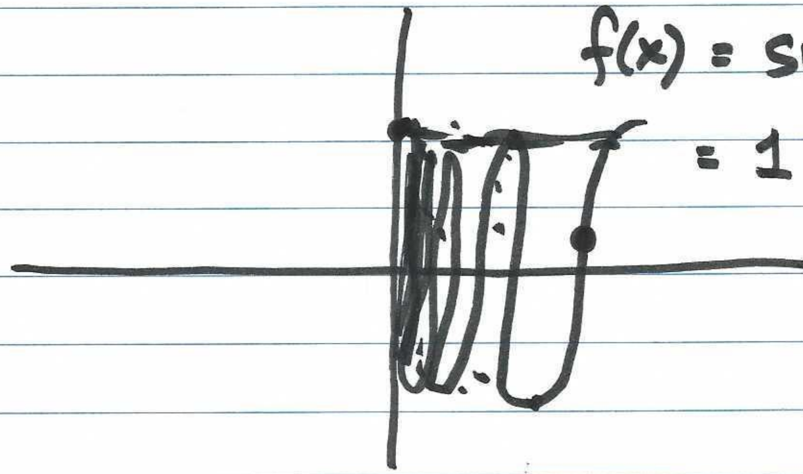


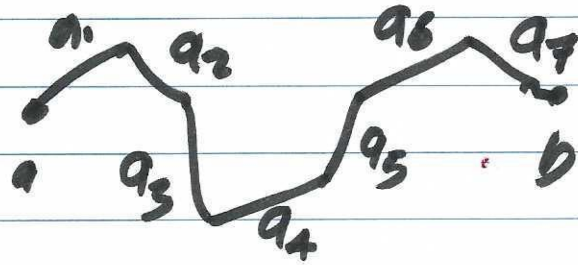
range disconnection induces same in domain

Contradiction proves claim.

Connected / not Pathwise Connected

$$f(x) = \sin\frac{1}{x} \quad x \neq 0$$
$$= 1 \quad x = 0$$





Let B be bounded in $\mathbb{R}^k$

Type 1 limit does not exist or $\neq f(x_0)$

Type 2 $\underbrace{\text{limit exists}}_{\text{removable}}$ but $\neq f(x_0)$

$$\hat{f}(x) = \begin{cases} f(x) & \text{if } x \neq x_0 \\ f(x_0) & \text{if so} \end{cases}$$

$$f(x) = \begin{cases} \dfrac{x^2-2}{x-2} & x \neq 2 \\ 4 & \text{if } x = 2 \end{cases}$$