Logic - Lecture 5

Tautology, Contradiction, & Contingency

A statement form is an expression like $(p \supset q) \equiv (q \vee \neg p)$ which has no semantic content.

A statement instance is a statement form where the variables that appear are given meaning.

Some statement forms are always true (relative to their major operator). These are called tautologies. Here is a truth table for the statement form above:

p	q	$(p \supset q)$	$\equiv$	$(q \vee \neg p)$
T	T	T	T	T
T	F	F	T	F
F	T	T	T	T
F	F	T	T	T

We will use this fact when we develop "replacement rules" for proving arguments.

p implies q <u>is the same</u> as q or not-p.

If a statement is a substitution instance (given meaning) of a tautology, it is also called a tautology. "It is what it is".

$I \equiv W$, or more properly, $I \equiv I$.

A <u>contradiction</u> is the opposite of a tautology - always false.

$\neg(p \equiv \neg q) \equiv \neg(p \equiv q)$ is such a statement form:

p	q	$\neg$	$(p \equiv \neg q)$	$\equiv$	$\neg$	$(p \equiv q)$
T	T	T	F	F	F	T
T	F	F	T	F	T	F
F	T	F	T	F	T	F
F	F	T	F	F	F	T

Note that the negation of a tautology is always a contradiction and vice versa. A substitution instance of a contradiction is also called a contradiction.

Finally, a <u>contingency</u> is a statement form that is neither a tautology or contradiction.

Here is an example: $(p \supset q) \cdot \neg p$

p	q	$(p \supset q)$	$\cdot$	$\neg p$
T	T	T	F	F
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Two statement forms are said to be <u>logically equivalent</u> whenever their truth tables are identical under the major operator.

p	q	$p \supset q$	$\neg p$	$\vee$	q
T	T	T	F	T	
T	F	F	F	F	
F	T	T	T	T	
F	F	T	T	T	

So $(p \supset q) \equiv (\neg p \vee q)$

It is even possible for two statements to be logically equivalent and have different numbers of variables. You can check $(\neg p \cdot \neg q) \equiv ((\neg p \cdot \neg q) \cdot r) \vee ((\neg p \cdot \neg q) \cdot \neg r)$. Two logically equivalent statements joined by the biconditional operator "$\equiv$" form a tautology. Some books use this as the definition of logical equivalence.

A statement form <u>logically implies</u> another if there is no row in their joint truth table where the first is true and the second is false.

Two statement forms which logically imply each other are logically equivalent.

A set of statements is <u>consistent</u> if (and only if) there is at least one row in their joint truth table in which every statement is true. See if you can show that ① $(p \supset (p \cdot q))$, ② $(\neg p \supset \neg(p \vee q))$, and 2 ③ $(\neg(p \cdot q) \equiv (\neg p \cdot \neg q)$ are mutually consistent. A set of statements is <u>inconsistent</u> if there is no row in their joint truth table where they all come out true.

What does that mean? The conjunction of all the statements is apparently a contradiction. A set of premises that is inconsistent is not a basis for an argument. Pro tip: a statement is a tautology if its truth table has all T's under the major operator. A set of statements is consistent if their combined table has a <u>row</u> of all T's.

We have studied four kinds of logic problems that are resolved with truth tables:

1. Validity of arguments
2. Tautology/Contradiction/Contingency
3. Logical Equivalence/Implication
4. Consistency

Here are some counterintuitive facts:

1. Inconsistent premises imply a valid argument [false $\supset$ anything is true]
2. An argument with a tautologous conclusion is valid [anything $\supset$ true is true]
3. If the conjunction of all premises implies the conclusion, the argument is automatically valid.
4. Two tautologies are equivalent and two contradictions are logically equivalent