Logic - Lecture 9

Conditional & Indirect Proofs

Two powerful methods:

Conditional Proof

Suppose we have an argument where the conclusion is a conditional statement $p \supset q$. We have our usual premises:

1) Premise 1

⋮

n) Premise n and we assume p and write it indented

n+1) p ← indent

⋮

then we use the premises and assumption to prove q.

n+k) q

n+k+1) $p \supset q$ ■

The double-headed arrow is our scope marker. No intermediate conclusions can be removed and used outside the scope to further the argument. All we are allowed to do is show $p \supset q$ and then discharge the scope to move on.

An example:

1)	$(A \vee B) \supset (C \cdot D)$	/∴ $A \supset C$
2)	A	C.P. ← assumption
3)	$A \vee B$	Add; 2
4)	$C \cdot D$	M.P.; 1,3
5)	C	Simp; 4
6)	$A \supset C$	C.P. ← conclusion

Indirect Proof (Proof by Contradiction)

One way to prove a statement is to assume its negation and work thru a derivation that produces the conjunction of some other statement and its negation. This is automatically a contradiction. We present this process much like C.P.:

1) Premise 1 $\quad / \therefore p$

⋮

n) Premise n

n+1) $\neg p$

⋮

n+k) $q \cdot \neg q$

n+k+1) p

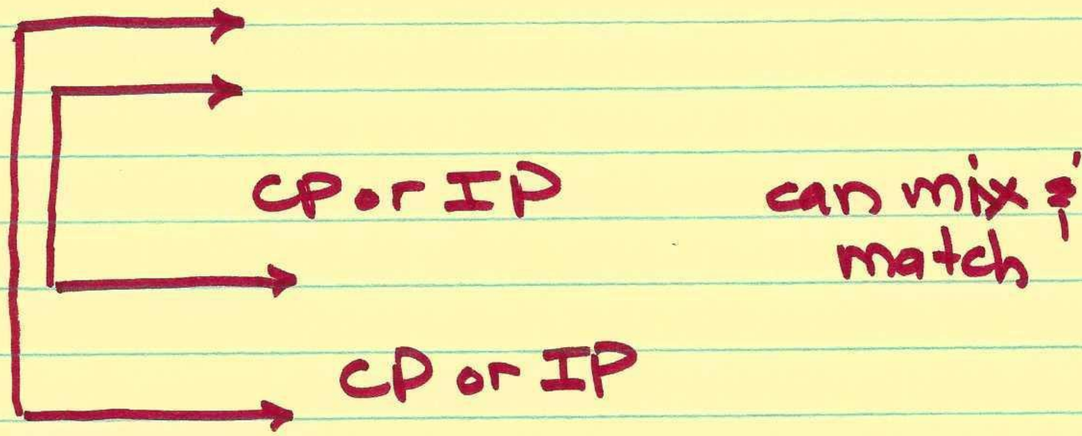
Once again, we have a sub-proof marked with the scope of the assumed negation. Upon obtaining the contradiction, the scope is discharged and the next line presents the un-negated statement that was purposely negated to start the sub-proof. An example:

Premises
1) $A \supset \neg(B \vee C)$
2) $(\neg B \vee \neg D) \supset F$
3) $F \supset \neg A$ $\quad / \therefore \neg A$

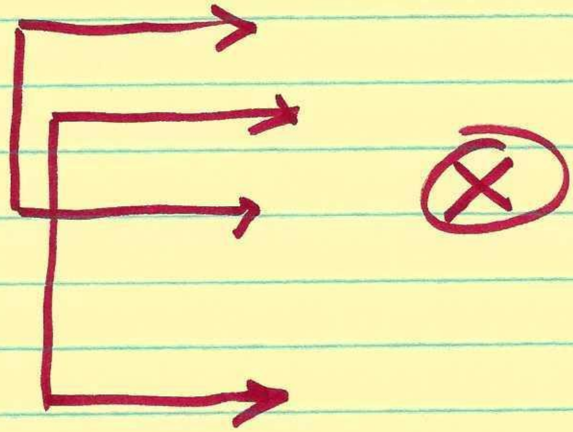
4) A IP (indirect proof assumption)
5) $\neg(B \vee C)$ MP; 1,4
6) $\neg B \cdot \neg C$ DeM; 5
7) $\neg B$ Simp; 6
8) $\neg B \vee \neg D$ Add; 7
9) F MP; 2,8
10) $\neg A$ MP; 3,9
11) $A \cdot \neg A$ Conj; 4,10

12) $\neg A$ IP; 4 to 11

There is no reason that these methods cannot be nested:



What cannot be done is using <u>anything</u> inside a scope outside of that scope. In particular, scopes cannot partially overlap:



Our inference rules and replacement rules are examples of logical __theorems__.

{Theorems} = {Tautologies}

Theorems are formulas derived without premises. Modus Ponens can be compressed to $[(p \supset q) \cdot p] \supset q$. If you do a

M.O.

truth table, there are all T's under the major operator. All T's means it is a tautology. We needed to establish __some__ inference and replacement rules by using truth tables in order to avoid circular reasoning. We can't prove MP by actually using it! For example, we could attempt a conditional proof:

1) $(p \supset q) \cdot p$ CP assumption

2) $p \supset q$ Simp; 1

3) p Simp; 1

4) q $\boxed{\text{MP; 2, 3}}$ circular!

5) $[(p \supset q) \cdot p] \supset q$ CP; 1-5

But once we have an inventory of rules (tautologies) we can prove theorems like this: $(p \cdot q) \supset p$

1) $p \cdot q$ CP assumption

2) p Simp; 1

3) $(p \cdot q) \supset p$ CP; 1-2

So new theorems, which have no initial premises, can be proven by:

1) Truth Tables - look for tautology
2) C.P.
3) I.P.
4) Chains of replacement

An example of 4) is: $p \supset q \equiv p \supset (q \vee r)$

$(p \supset q) \equiv \neg p \vee q \equiv (\neg p \vee q) \vee r \equiv$
$\neg p \vee (q \vee r) \equiv p \supset (q \vee r)$

An example of 2) is:

$(p \supset (q \supset r)) \supset (q \supset (p \supset r))$

which is not "obvious".

1)	$p \supset (q \supset r)$	CP assumption
2)	q	CP assumption
3)	p	CP assumption
4)	$q \supset r$	MP; 1, 3
5)	r	MP; 2, 4
6)	$p \supset r$	CP; 3-5
7)	$q \supset (p \supset r)$	CP; 2-6
8)	$(p \supset (q \supset r)) \supset (q \supset (p \supset r))$	CP; 1-7

Another way to conclude an I.P. :

$\neg p$

$\vdots$

$q \cdot \neg q$

$q \vee p$ Addition

$\neg q$ Simp

p D.S.