Logic - Lecture 11

Quantifiers & Categorical Propositions

Recall that a propositional function is a singular sentence (something specific has a name and a property) with the name(s) removed and variables substituted for them:

x is large, y is pretty, and so forth.

A quantifier is an operator placed in front of a propositional function that tells us for how many things a propositional function is true.

The phrase "x is large" has no truth value because we don't know what x is. If x = "elephant" - true. If x = "mouse" - false. There are two types of quantifier: "some" or "all". "Some" is the existential quantifier. We indicate it with the symbol $\exists$. The line $(\exists x) Lx$ means "some x is large" (L = "large") or "there exists an x that is large. This existential quantifier creates an existential sentence. Note that it has truth value. There may or may not be a large thing.

"All" is the universal quantifier. It is symbolised by (x). In math you may see it as $\forall$. If we want to say everything is large, we write $(x)\,Lx$. This is a universal sentence.

With obvious notation:

$(\exists x)\,Mx$	There is a mystery
$(\exists y)\,Ty$	There are many translations of the Bible
$(\exists x)\,Dz$	There is at least a dollar in my pocket
$(\exists w)\,Bw$	Black holes exist.
$(x)\,Ux$	Everything is unique
$(y)\,By$	Anything is better than broccoli
$(\exists z)\,Iz$	Some man is an island
$\neg(\exists z)\,Iz$	No man is an island
$(z)\,\neg Iz$	No man is an island

How would we encode "Something dug up the lawn"? It seems like we could write Ds, a simple sentence with D = "dug up the lawn" and s = "something". But this is an error. Something is not a name, but an indication of existence. The correct translation would be $(\exists x)Dx$.

The <u>scope</u> of a quantifier is defined the same way as the scope of a negation. The quantifier applies to the first complete formula written immediately after it.

So we have $(x)\overbrace{Fx}^{\text{scope}}$ but in the term

$\underbrace{(x)\,Fx}_{\text{scope}} \supset Gy$, Gy is <u>outside</u> the

scope of (x). Contrast this with

$(x)\underbrace{[F(x) \supset G(y)]}_{\text{scope}}$.

A <u>bound</u> variable is one that falls within the scope of its own quantifier. A <u>free</u> variable does not.

This can be tricky:

$(x)\,Fx \supset Gy$	x bound y free
$(x)\,Fx \supset Gx$	1st x bound 2nd x free
$(x)\,[Fx \supset Gy]$	x bound y free
$(x)\,[Fx \supset Gx]$	both x's bound

Negated Quantifiers

There are some philosophical land mines here and there. If we say "nothing exists" and read it as a singular sentence, then it is "nothing" that exists ... and that's something after all. We handle these as negated existential sentences. "Nothing will help" is coded as $\neg(\exists x)Hx$, where H means "will help". Here are the equivalences among negated quantifiers:

ϕ is some predicate and x is the variable:

Quantifier Negation (QN) Equivalences

1. $\neg(\exists x)\phi x \equiv (x)\neg\phi x$
2. $\neg(x)\phi x \equiv (\exists x)\neg\phi x$
3. $\neg(\exists x)\neg\phi x \equiv (x)\phi x$
4. $\neg(x)\neg\phi x \equiv (\exists x)\phi x$

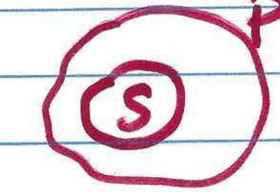
Categorical Propositions

In a singular sentence, the subject is a named individual or thing. In a categorical sentence, the subject and predicate are classes and the sentence states some inclusion or exclusion relation between them. The subject expression of a sentence is a noun with all its modifiers. The <u>predicate expression</u> is the sentence minus the subject expression and any quantifiers.

We have four possible situations:

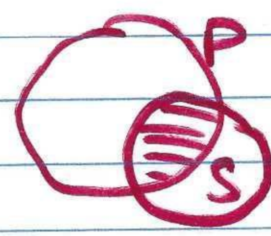
Let S be a subject class and P be a predicate class.

1. All S are P



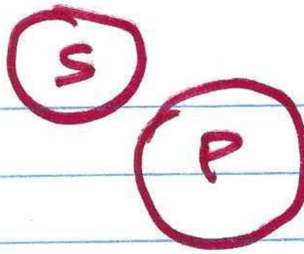
This is total inclusion. We call this type of sentence a universal affirmative and abbreviate the type as "A".

2. Some S are P



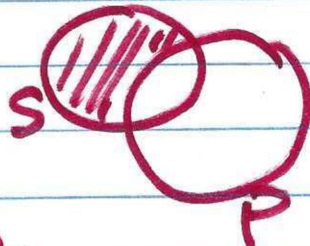
This is partial inclusion.

We call it a particular affirmative and say it is a type "I" sentence.



3 No S are P

This is total exclusion. We call it universal negative. It is an "E" sentence.



4 Some S are not P

This is partial exclusion, called a particular negative. This is an "O" type sentence.

A mnemonic: (from Latin)

AFFIRMO – I affirm

universal particular

NEGO – I deny

universal particular

Symbolizing Categorical Statements

Let ϕ and ψ be two predicates that are classes.

1. A typical A sentence would be

$$(x)(\phi x \supset \psi x)$$

Example: ϕ = "is a bear"
ψ = "is a ferocious animal"

$(x)(\phi x \supset \psi x)$ means if x is a bear then x is ferocious, or "all bears are ferocious".

The negation of this sentence is $(\exists x)(\phi x \cdot \neg \psi x)$. "Some bear is not ferocious. So "$\neg$A" is an "O".

$$\neg (x)(\phi x \supset \psi x) \equiv (\exists x)(\phi x \cdot \neg \psi x)$$

2 A typical I sentence would be

$(\exists x)(\phi x \cdot \psi x)$.

Example: ϕ and ψ as previously:

"There is a bear that is ferocious".

Negating this would be "All bears are not ferocious." So "$\neg$ I" is an "E".

$$\neg(\exists x)(\phi x \cdot \psi(x)) \equiv (x)(\phi x \supset \neg \psi x)$$

3 A typical E sentence would be

$(x)(\phi x \supset \neg \psi x)$ as in 2. Its negation is $(\exists x)(\phi x \cdot \psi x)$. So "$\neg$ E" is an "I"

$$\neg(x)(\phi x \supset \neg \psi x) \equiv (\exists x)(\phi x \cdot \psi x)$$

"It is not the case that all bears are

not ferocious" is the same as "some bears are ferocious".

4 A typical O sentence is $(\exists x)(\phi x \cdot \neg \psi x)$. Negated this becomes an A sentence:

$$\neg(\exists x)(\phi x \cdot \neg \psi x) \equiv (x)(\phi x \supset \psi x).$$

These are the categorical quantifier negation (CQN) equivalences.

Aristotle's way of organizing all of this