(2.4) Exponential Functions

$2^2 = 4 \quad 3^4 = 81$

2^x or 10^y are exponential functions

General situation b is a base

$f(x) = b^x \quad b \neq 0 \quad b \neq 1 \quad b > 0$

$$\left.\begin{array}{l}(-2)^1 = -2 \\ (-2)^2 = +4 \\ (-2)^3 = -8\end{array}\right\} \text{oscillation}$$

1) $b^x \cdot b^y = b^{x+y}$ $\quad \underbrace{b \cdot b \cdots b}_{x} \cdot \underbrace{b \cdot b \cdot b \cdots b}_{y}$

2) $b^x / b^y = b^{x-y}$ $\quad \dfrac{b \cdot b \cdot b \cdots b \leftarrow x}{b \cdot b \cdot b \cdots b \leftarrow y}$

$\dfrac{1}{b^{y-x}} = b^{x-y}$

3) $(b^x)^y = b^{xy} = (b^y)^x$

$$y \text{ rows} \left\{\begin{array}{ll} b \cdot b \cdots b & x \\ b \cdot b \cdots b & x \\ \vdots & \\ b \cdot b \cdots b & x \end{array}\right.$$

We write $\sqrt[y]{b} = b^{1/y}$

$$\sqrt[y]{b^x} = b^{x/y}$$

4) So $b^{x/y} = \sqrt[y]{b^x} = \left[\sqrt[y]{b}\right]^x$

$\sqrt{2} = 2^{1/2}$ $\quad \sqrt{32} = \sqrt{2^5} = \underline{2^{5/2}}$

Ex. $\sqrt[3]{4} \cdot \sqrt[7]{64}$

$$\downarrow \qquad \downarrow$$

$$\underline{\sqrt[3]{2^2} \cdot \sqrt[7]{2^6}} = 2^{2/3} \cdot 2^{6/7} = 2^{\left(\frac{2}{3}+\frac{6}{7}\right)}$$

$$\left(\frac{2}{3}+\frac{6}{7}\right) = \frac{14+18}{21} = \frac{32}{21} = 2^{32/21}$$

Solving exponential equations:

$$b^x = A \qquad 9^x = 27 \rightarrow 3^{2x} = 3^3$$

$$\Rightarrow 2x = 3$$

or

$$x = 3/2$$

Fundamental Principle

$$b^x = b^y$$

$$x = y$$

$32 = 2^5$
$64 = 2^6$
$128 = 2^7$

Ex: $32^{2x-1} = 128^{x+3}$

$$(2^5)^{2x-1} = (2^7)^{x+3}$$

$$2^{10x-5} = 2^{7x+21}$$

$$\Rightarrow 10x-5 = 7x+21$$

$$3x = 26$$

$$x = 26/3$$

Simple Interest vs Compound Interest

$I = Prt$ — P: principa, r: rate, t: time

in years = per annum

$P = \$2500$
$r = 6\frac{5}{8}\%$ p.a.
$t = 3$ yrs
$I = \$500^{63}$

Compound Interest

r is simple rate/yr

yr	P_{beg}	I	P_{end}
1	P	rP	$P+rP$
2	$P+rP$	$r(P+rP)$	$(P+rP)(1+r)$
3	$P(1+r)^2$	$rP(1+r^2)$	$P(1+r)^3$
4	$\vdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	
n			$P(1+r)^n$

A is <u>future</u> value

P is <u>present</u> value

"growth factor"

$$A = P(1+r)^n$$

$$A = P\left(1+\frac{r}{12}\right)^{12n}$$

m : # months

n = # years

$m = 12n$

Continuous Compounding:

$A = \underline{Pe^{rt}}$ $\quad e \approx 2.71828\ldots$

$P = \$1$

how much after @ 5% p.a.

$A = 1e^{(.05)(1)} = \underline{\$1.05127}$

\$1 @ 5% simple interest

invested 0 A.D. $\quad \$101^{20}$

\$1 @ 5% cont. comp.

invested 0 A.D.

$A = Pe^{(.05)(2024)} = 1 \cdot e^{101.2}$

$= 8.924 \times 10^{43}$

Rule of 70 (actually 69.3)

Want money to double in x years

We need $^{70}/_{x}$ % cont. comp. int.

World Popotation modeled by :

$$P(t) \approx 3184 e^{0.0158t}$$

$$P(2024) \approx 3184 e^{1.0112} = 8752.5 \times 10^6$$

Logarithms

$$b^x = A \iff \log_b A = x$$

$$b^{\log_b A} = A$$

$b = 10$ common $\log A$

$b = e$ natural $\ln A$

$b = 2$ binary $\lg A$

all others need subscript $\log_5 A$

$\log 2 \approx .30103$ +
$\log 3 \approx .47712$
$\log 6 \approx .77815$

Rules for logs:

1) $\log_b A + \log_b B = \log_b(AB)$

2) $\log_b A - \log_b B = \log_b(A/B)$

3) $n \cdot \log_b A = \log_b(A^n)$

4) $\dfrac{\log_b A}{n} = \log_b A^{1/n} = \log_b \sqrt[n]{A}$

$x = \log_b A \qquad y = \log_b B$

$b^x \cdot b^y = b^{x+y}$

$b^{\log_b A} \cdot b^{\log_b B} \qquad b^{\log_b(AB)}$

$f(x) = \log_b x$

Change-of-base

Given $\log_b A$ & I want $\log_c A$

$$\log_c A = \frac{\log_b A}{\log_b c}$$ p.103

$\log x + \log(x-3) = 1$

$\log x(x-3) = 1$

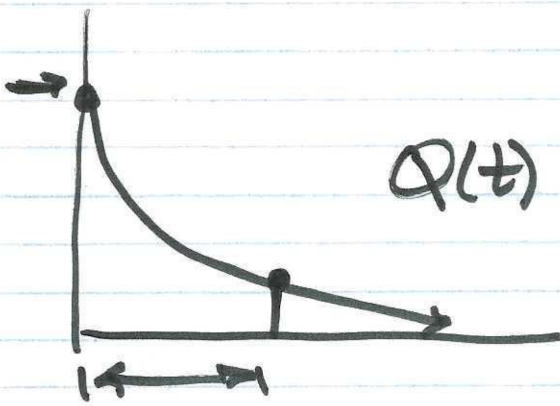
$10^{\log(x^2-3x)} = 10^1$

$x^2 - 3x = 10$

$x^2 - 3x - 10 = 0$

$x = \frac{3 \pm \sqrt{9+40}}{2}$ $\quad x = \frac{3 \pm 7}{2}$

$x = 5, -2$



$$Q(t) = Q_0 e^{-\lambda t}$$