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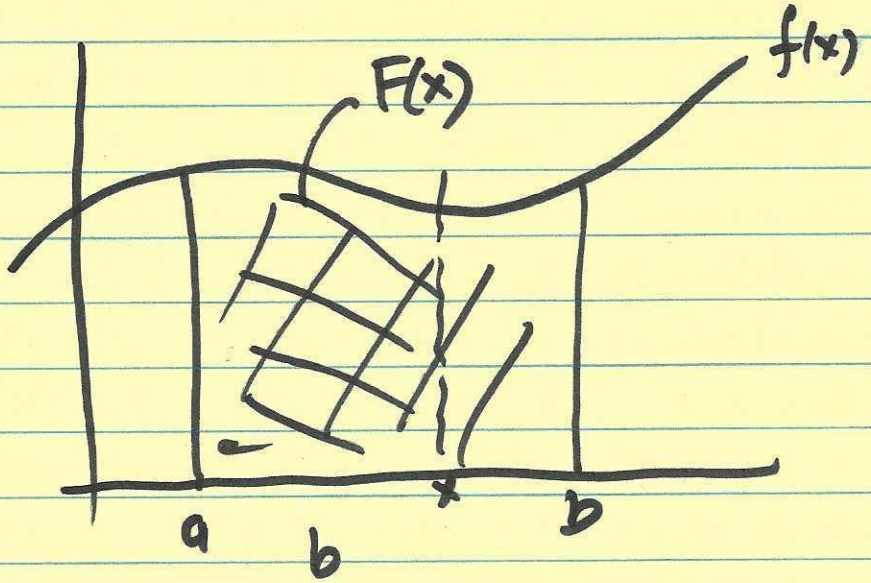
Integration Using Substitutions



$$\text{Area} = \int_a^b f(x)\,dx$$

Area is really the Function F(x)

$$\text{Amount that has occurred} = \int (\text{rate})\, d(\text{time})$$

Typical integral (hard to do)

$$\int x e^{x^2} dx \qquad \rightarrow e^u$$

Let $u = x^2$ $\quad \frac{du}{dx} = 2x \Rightarrow du = 2x\,dx$

$$\int x e^{x^2} dx = \int e^u \frac{du}{2} = \frac{1}{2}\int e^u du = \frac{1}{2}e^u \rightsquigarrow$$

$$\frac{1}{2}e^u = \boxed{\frac{1}{2}e^{x^2}}$$

~~$\int (2x^3+1)^4 dx$~~ $\int (2x^3+1)^4 (6x^2 dx)$

$u = 2x^3 + 1$

$\frac{du}{dx} = 6x^2 + 0$

$du = 6x^2 dx$

(3)

$$\int (2x^3+1)^4 6x^2\,dx \rightsquigarrow \int u^4\,du = \frac{u^5}{5} + C$$

$$\hookrightarrow \frac{(2x^3+1)^5}{5}$$

$$\int x^3\sqrt{3x^4+10}\,dx$$

Try: $u = 3x^4+10$

$du = 12x^3\,dx \Rightarrow dx = \frac{du}{12x^3}$

$$\int \sqrt{u}\cdot\frac{du}{12x^3} = \frac{1}{12}\int \sqrt{u}\,du$$

$$\frac{1}{12}\int u^{1/2}\,du$$

$$\frac{1}{2}\int u^{1/2}\,du = \frac{1}{2}\frac{u^{3/2}}{3/2} = \frac{1}{2}\cdot\frac{2}{3}u^{3/2} = \frac{u^{3/2}}{3}$$

Back-sub: $\dfrac{(3x^4+10)^{3/2}}{3}$

$$\int \frac{x+1}{(4x^2+8x)^3}\,dx$$

$u = 4x^2+8x \Rightarrow du = (8x+8)\,dx$

$\downarrow$

$8(x+1)\,dx$

$$\int \frac{1}{u^3}\frac{du}{8} = \frac{1}{8}\int u^{-3}\,du = \frac{1}{8}\frac{u^{-2}}{-2} \Rightarrow$$

$$\boxed{-\frac{1}{16}\cdot\frac{1}{u^2}}$$

(5)

$$\int \underset{\frac{2}{3}}{x}\sqrt{1-x}\,dx$$

Try: $\underline{u = 1-x}$ $\quad du = -dx$

$$\int x\sqrt{u}\,(-du)$$

$x = 1-u$

$$= \int (1-u)u^{1/2}(-du) = \int (u-1)u^{1/2}du$$

$$= \int (u^{3/2} - u^{1/2})du = \boxed{\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}}$$

$$\rightarrow \frac{2}{5}(1-x)^{5/2} - \frac{2}{3}(1-x)^{3/2}$$

(b)

$$f(x) = \frac{A}{B+x}$$

$f(x)$ is rate of change of a song on Hot 100

x is current position

A, B are constants

$F(x) = \int f(x)\,dx$ is popularity index

Find $F(x) = \int \frac{A\,dx}{B+x} = A\int \frac{dx}{B+x}$

Let $u = B+x \Rightarrow du = dx$

$$A\int \frac{du}{u} = A\ln u + C$$

$$= \boxed{A\ln(B+x)+C}$$

#6 $\int \sqrt{1-x}\, dx$

$u = 1-x \Rightarrow x = 1-u$

$du = -dx$

$\int \sqrt{u} \cdot (-du) = -\int u^{1/2} du = -\frac{2}{3}u^{3/2}$

#8 $\int 4x^3 e^{x^4} dx$

$u = x^4 \quad du = 4x^3 dx$

$\int e^u du = e^u \rightarrow$

e^{x^4}

#22 $\int \frac{e^{\sqrt{y}}}{2\sqrt{y}}\,dy = ?$

Try: $u = \sqrt{y} = y^{1/2} \Rightarrow du = \frac{1}{2}y^{-1/2}dy = \frac{dy}{2\sqrt{y}}$

$\int \frac{e^u 2\sqrt{y}\,du}{2\sqrt{y}}$ $\qquad dy = 2\sqrt{y}\,du$

$\int e^u du = e^u \rightarrow \boxed{e^{\sqrt{y}}}$

#21 $\int \frac{e^{1/z}}{z^2}\,dz$

Try: $u = \frac{1}{z} \Rightarrow du = -\frac{1}{z^2}dz$

$\int e^u \cdot (-du) = -e^u \rightsquigarrow \boxed{-e^{1/z}}$

(9)

$$P'(t) = te^{-t^2}$$

$$P(t) = \int P'(t)\,dt = \int te^{-t^2}\,dt$$

Try: $u = t^2 \quad du = 2t\,dt$

$$\int e^{-u}\left(\frac{du}{2}\right) = \frac{1}{2}\int e^{u}\,du = \frac{1}{2}(-e^{-u}) + C$$

$$P(t) = -\frac{1}{2}e^{-t^2} + C$$

$$P(3) = -\frac{1}{2}e^{-3^2} + C = 10{,}000$$

$$= C = 10{,}000 + \frac{e^{-3}}{2} = 10{,}011$$

$$\boxed{P(t) = 10{,}011 - \frac{1}{2}e^{-t^2}}$$

(10)

$N(0) = 37$

$$N'(t) = \frac{100t}{t^2+2}$$

$N(t)$ = # people infected @ t days

$$N(t) = \int \frac{100t\,dt}{t^2+2}$$

Let $u = t^2+2 \Rightarrow du = 2t\,dt$

$$\int \frac{50\,du}{u} = 50\int \frac{du}{u} = 50 \ln u$$

$$N(t) = 50\ln(t^2+2) + C$$

$$N(0) = 50\ln(2) + C = 37$$

$$= C = 37 - 50(.693)$$

$$\approx 2$$

$$\boxed{N(t) = 50\ln(t^2+2)+2}$$

(11)

$$N(21) = 50 \ln(443) + 2$$