Acceleration

$$\mathbf{r}(t) \rightsquigarrow \mathbf{r}'(t) = \mathbf{v}(t)$$

$$\mathbf{v}(t) = \frac{d\mathbf{r}}{ds}\frac{ds}{dt} = \mathbf{T}\frac{ds}{dt}$$

($\frac{d\mathbf{r}}{ds} = \mathbf{T}$)

$$\mathbf{a}(t) = \mathbf{v}'(t) = \frac{d}{dt}\left(\mathbf{T}\frac{ds}{dt}\right) = \frac{d}{dt}\left(\frac{ds}{dt}\right)\mathbf{T} + \frac{ds}{dt}\frac{d\mathbf{T}}{dt}$$

Chain Rule: $\frac{d\mathbf{T}}{dt} = \frac{d\mathbf{T}}{ds}\frac{ds}{dt}$

$$= \frac{d^2s}{dt^2}\mathbf{T} + \frac{ds}{dt}\left(\frac{d\mathbf{T}}{ds}\frac{ds}{dt}\right)$$

$$= \frac{d^2s}{dt^2}\mathbf{T} + \left(\frac{ds}{dt}\right)^2\frac{d\mathbf{T}}{ds} = \frac{d^2s}{dt^2}\mathbf{T} + \left(\frac{ds}{dt}\right)^2(\kappa\mathbf{N})$$

$$\boxed{\mathbf{a}(t) = \frac{d^2s}{dt^2}\mathbf{T} + \kappa\left(\frac{ds}{dt}\right)^2\mathbf{N}}$$

difficult

$$\mathbf{a}(t) = a_T\mathbf{T} + a_N\mathbf{N}$$

$$a_N = \sqrt{|\mathbf{a}|^2 - a_T^2}$$

Ex 1 $\quad t \geq 0$

$$r(t) = (\cos t + t\sin t)\,\hat{i} + (\sin t - t\cos t)\,\hat{j}$$

Want to write acceleration as:

$$a(t) = a_T T + a_N N$$

$$v(t) = (-\sin t + \sin t + t\cos t)\,\hat{i} + \ldots$$

$$= (\cos t - \cos t + t\sin t)\,\hat{j}$$

$$v(t) = t\cos t\,\hat{i} + t\sin t\,\hat{j}$$

$$|v(t)| = \sqrt{t^2\cos^2 t + t^2\sin^2 t} = t(1) = t$$

$$a_T = \frac{d}{dt}(t) = 1$$

✓ Recall $a_N = \sqrt{|a|^2 - a_T^2}$

$$a(t) = (\cos t - t\sin t)\,\hat{i} + (\sin t + t\cos t)\,\hat{j}$$

$$|a(t)| = \sqrt{\cos^2 t - 2t\cos t\sin t + t^2\sin^2 t + \ldots}$$

$$\ldots \sqrt{\sin^2 t + 2t\cos t\sin t + t^2\cos^2 t}$$

$$= \sqrt{\cos^2 t + \sin^2 t + t^2(\sin^2 t + \cos^2 t)}$$

$$|a(t)| = \sqrt{1 + t^2}$$

$$\therefore a_N = \sqrt{(1+t^2) - 1} = \sqrt{t^2} = t$$

So acceleration is $a_T T + a_N N =$

$$= T + tN$$

Practical Calculations

$$T = \frac{\mathbf{v}}{|\mathbf{v}|}.$$

$$N = \frac{T'(t)}{|T'(t)|}.$$

$$B = T \times N.$$

$$\kappa = \frac{|\mathbf{v} \times \dot{\mathbf{v}}|}{|\mathbf{v}|^3}. \qquad \text{"}\cdot\text{" is time derivative}$$

$$\tau = \frac{\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix}}{|\mathbf{v} \times \dot{\mathbf{v}}|^2}$$

$$\mathbf{a} = \dot{\mathbf{v}} = a_T T + a_N N$$

$$a_T = \frac{d}{dt}|\mathbf{v}| \qquad a_N = \kappa\left[s'(t)\right]^2 = \kappa |\mathbf{v}|^2$$

Space curve $\mathbf{r}(t) = x(t)\,\hat{\imath} + y(t)\,\hat{\jmath} + z(t)\,\hat{k}$

Velocity $\mathbf{v}(t) = \mathbf{r}'(t)$ (defn)

Speed $|\mathbf{v}(t)| = s'(t)$ (defn)

Distance along curve $s(t) = \int_{t_0}^{t} |\mathbf{v}(t)|\,dt$ (defn)

Unit tangent $\mathbf{T} = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \mathbf{r}'(s)$ ← (defn)

Unit normal $\mathbf{N} = \dfrac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \dfrac{1}{\kappa}\,\mathbf{T}'(s)$ ← (defn)

Unit binormal $\mathbf{B} = \mathbf{T} \times \mathbf{N}$ (defn)

Curvature $\kappa = \dfrac{|\mathbf{T}'(t)|}{|\mathbf{v}(t)|} = |\mathbf{T}'(s)|$ ← (defn)

Torsion $\tau = -\mathbf{B}'(s) \cdot \mathbf{N}$ (defn)