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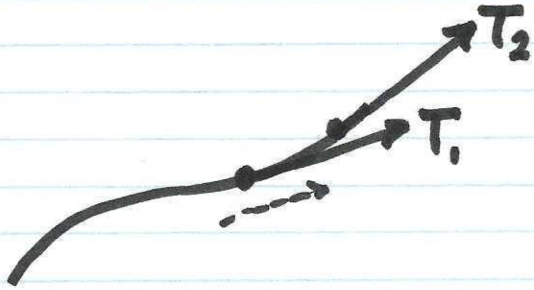
Space Curves - continued

Curvature parameter κ (kappa)

Def^n: $\kappa := \left| \frac{dT}{ds} \right|$



Alternative def^n $\kappa = \frac{1}{|v|} \left| \frac{dT}{dt} \right|$

$T(t) = \frac{v(t)}{|v(t)|}$

★ κ is 0 for straight path

Ex: $r(t) = a\cos t\,\hat{i} + a\sin t\,\hat{j}$

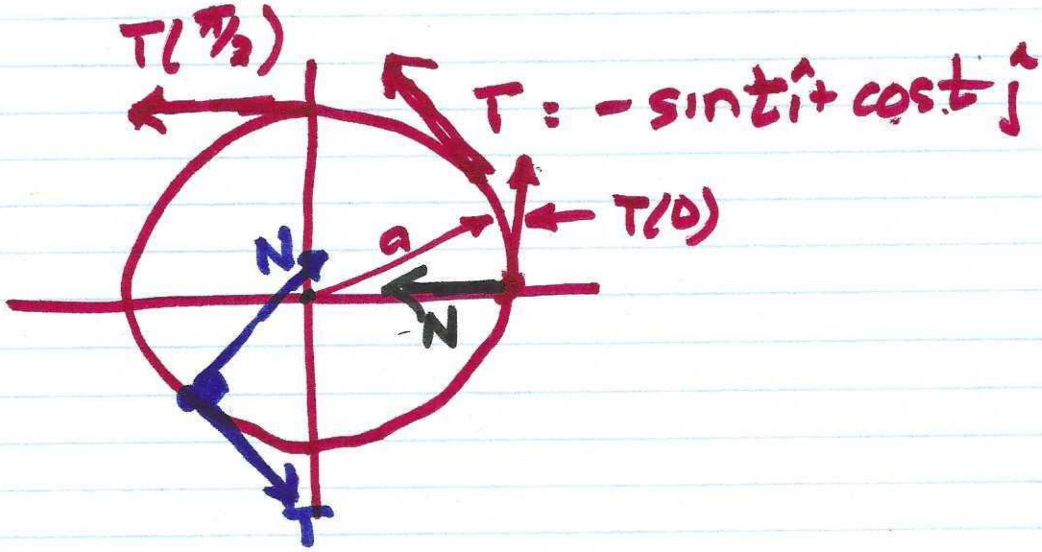
$v(t) = r'(t) = -a\sin t\,\hat{i} + a\cos t\,\hat{j}$

$|v(t)| = \sqrt{a^2\sin^2 t + a^2\cos^2 t} = a$

$\kappa = \frac{1}{a} \left| \frac{dT}{dt} \right|$

Unit tangent is: $T = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = ?$

$$\frac{-a\sin t\,\hat{i} + a\cos t\,\hat{j}}{a} = -\sin t\,\hat{i} + \cos t\,\hat{j}$$



$$\frac{dT}{dt} = ? = -\cos t\,\hat{i} - \sin t\,\hat{j}$$

$$\left|\frac{dT}{dt}\right| = \sqrt{\cos^2 t + \sin^2 t} = 1$$

So... $\kappa = \frac{1}{a} \cdot 1 = \frac{1}{a}$

Def^{n} the radius of curvature of $\mathbf{r}(t)$ is the reciprocal of the curvature usually denoted by ρ

So $\boxed{\kappa\rho = 1}$

Unit normal vector N

$$N = \frac{1}{\kappa}\frac{dT}{ds} = \frac{dT/ds}{|dT/ds|}$$

$$N = \frac{\frac{dT}{ds}}{\left|\frac{dT}{ds}\right|} \underbrace{\frac{\frac{dt}{ds}}{\frac{dt}{ds}}}_{1}$$

$\frac{ds}{dt} = 0$?
never true

$$N = \frac{\frac{dT}{\cancel{dt}}\frac{\cancel{dt}}{ds}}{\left|\frac{dT}{\cancel{dt}}\frac{\cancel{dt}}{ds}\right|} = \frac{dT}{ds}\Big/\left|\frac{dT}{ds}\right| = N$$

$$N = \frac{\frac{dT}{dt}}{\left\|\frac{dT}{dt}\right\|}$$ ← norm

Ex: $\mathbf{r}(t) = \cos 2t\,\hat{i} + \sin 2t\,\hat{j}$

Find $\mathbb{T}$ & $\mathbb{N}$

$$\mathbb{T} = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{-2\sin 2t\,\hat{i} + 2\cos 2t\,\hat{j}}{\sqrt{4\sin^2 2t + 4\cos^2 2t}}$$

$$= -\sin 2t\,\hat{i} + \cos 2t\,\hat{j}$$

$$\mathbb{N} = \frac{d\mathbb{T}/dt}{|d\mathbb{T}/dt|} = \frac{-2\cos 2t\,\hat{i} - 2\sin 2t\,\hat{j}}{\sqrt{4\cos^2 2t + 4\sin^2 2t}}$$

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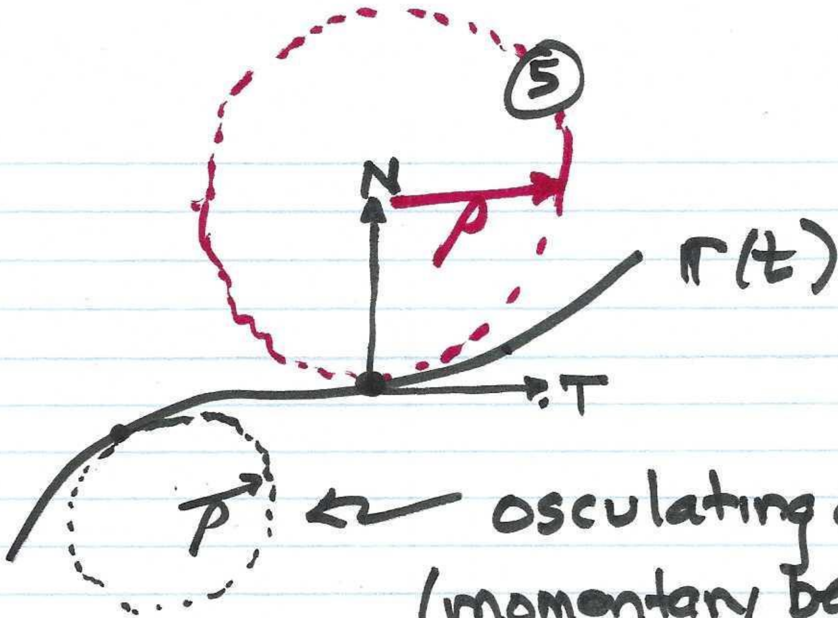
$$\mathbb{N} = -\cos 2t\,\hat{i} - \sin 2t\,\hat{j}$$

Advance warning:

Unit binormal vector $\mathbb{B} := \mathbb{T} \times \mathbb{N}$

$\kappa\rho = 1$



osculating circle
(momentary best approximation to $\mathbf{r}(t)$)

osculating circle lies in the osculating plane

$$y = x^2$$

Ex: Given $\mathbf{r}(t) = t\hat{\imath} + t^2\hat{\jmath}$

Want <u>osculating</u> circle @ $t=0$

Find $\mathbf{v}(t) = \mathbf{r}'(t) = \hat{\imath} + 2t\hat{\jmath}$

$$|\mathbf{v}(t)| = \sqrt{1+4t^2}$$

$$\mathbf{T} = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{1}{\sqrt{1+4t^2}}\left(\hat{\imath} + 2t\hat{\jmath}\right)$$

$$\cancel{\frac{d\mathbf{T}}{dt}} = \frac{1}{\sqrt{1+4t^2}}\hat{\imath} + \frac{2t}{\sqrt{1+4t^2}}\hat{\jmath}$$

$$= (1+4t^2)^{-1/2}\hat{i} + (2t)(1+4t^2)^{-1/2}\hat{j}$$

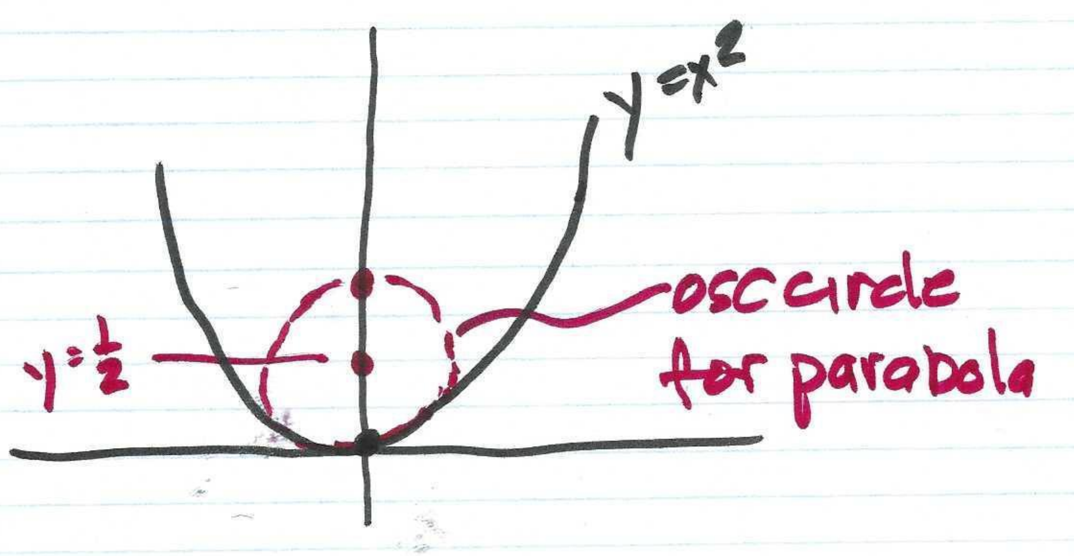
$$-\frac{1}{2}(1+4t^2)^{-3/2}\cdot(8t)\,\hat{i} + $$

$$\left[2(1+4t^2)^{-1/2} + (2t)\left(-\frac{1}{2}(1+4t^2)^{-3/2}\cdot(8t)\right)\right]\hat{j}$$

$$\frac{dT(0)}{dt} = -\underbrace{\cancel{4t}}_{0}\,\hat{i} + (2+0)\hat{j} = 2\hat{j}$$

So... $\frac{dT}{dt}$ eval @ 0 is $2\hat{j}$. $|2\hat{j}| = 2$

$\cancel{\frac{1}{2}}$ $\rho = \frac{1}{\kappa} = \frac{1}{2}$



<u>Helix</u>

$$\mathbf{r}(t) = a\cos t\,\hat{i} + a\sin t\,\hat{j} + bt\,\hat{k}$$

$$a, b \geq 0 \;;\; a^2 + b^2 \neq 0$$

Find $\mathbf{T}$:

$$\mathbf{r}'(t) = -a\sin t\,\hat{i} + a\cos t\,\hat{j} + b\hat{k}$$

$$|\mathbf{r}'(t)| = \sqrt{a^2\sin^2 t + a^2\cos^2 t + b^2}$$

$$= \sqrt{a^2(\sin^2 t + \cos^2 t) + b^2}$$

$$= \sqrt{a^2 + b^2}$$

$$\mathbf{T} = \frac{1}{\sqrt{a^2+b^2}}\left(-a\sin t\,\hat{i} + a\cos t\,\hat{j} + b\hat{k}\right)$$

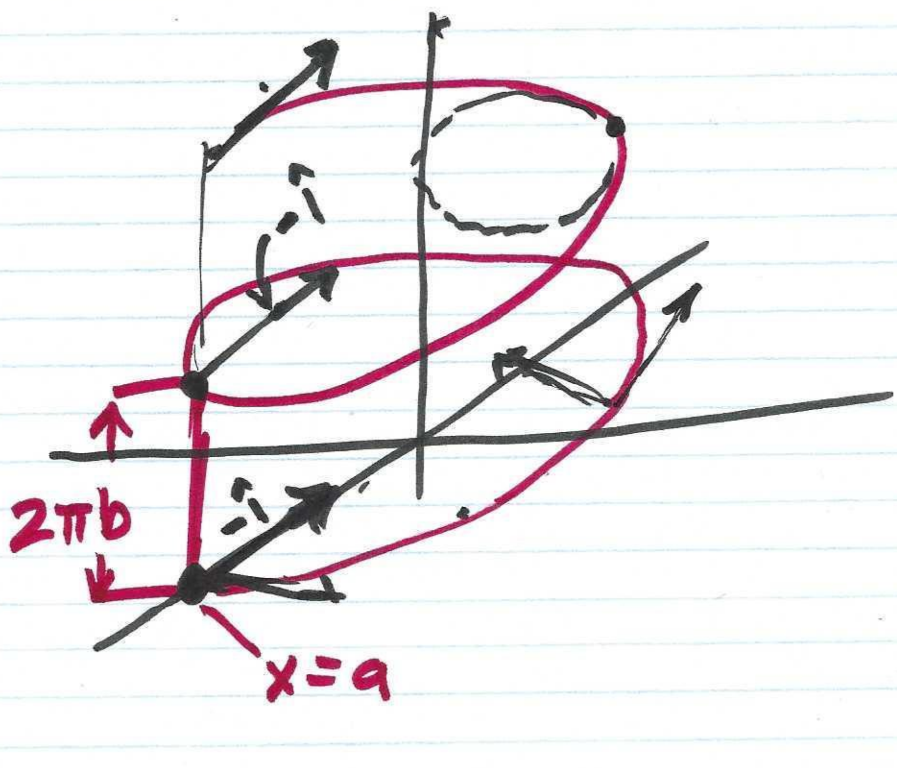
$$\kappa = \frac{1}{|\mathbf{r}'(t)|} \cdot \left|\frac{d\mathbf{T}}{dt}\right|$$

$$\frac{1}{\sqrt{a^2+b^2}}\left(\underline{-a\cos t}\;\hat{i} - \underline{a\sin t}\;\hat{j}\right)$$

$$K = \underbrace{\frac{1}{\sqrt{a^2+b^2}} \cdot \frac{1}{\sqrt{a^2+b^2}}}_{\frac{1}{a^2+b^2}} \left(\sqrt{a^2\cos^2 t + a^2\sin^2 t}\right)$$

$$\sqrt{a^2\cos^2 t + a^2\sin^2 t} \downarrow a$$

$$\text{So } K = \frac{a}{a^2+b^2} \Rightarrow \rho = \frac{a^2+b^2}{a}$$



$$\left|\frac{dT}{dt}\right| = \frac{1}{\sqrt{a^2+b^2}}\overbrace{\sqrt{a^2\cos^2 t + a^2\sin^2 t}}^{a}$$

$$= \frac{a}{\sqrt{a^2+b^2}}$$

$$N = \frac{dT/dt}{|dT/dt|} = -\frac{\sqrt{a^2+b^2}}{a}\left(\frac{a\cos t\,\hat{i} + a\sin t\,\hat{j}}{\sqrt{a^2+b^2}}\right)$$

$$= -\frac{(a\cos t\,\hat{i} + a\sin t\,\hat{j})}{a}$$

$$= \frac{-a\cos t\,\hat{i} - a\sin t\,\hat{j}}{a} =$$

$$N = -\cos t\,\hat{i} - \sin t\,\hat{j}$$

1. Given $\mathbf{r}(t) = t\,\hat{i} + \ln\cos t\,\hat{j}$

$-\frac{\pi}{2} \le t \le \frac{\pi}{2}$

Find $\mathbf{T}, \mathbf{N}, \kappa$

(i) $\mathbf{T} = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \dfrac{\hat{i} + (-\tan t)\,\hat{j}}{\sqrt{1+\tan^2 t}} = ?$

So... $\mathbf{T} = \dfrac{\hat{i} - \tan t\,\hat{j}}{\sec t}$ $\quad \dfrac{\tan t}{\sec t}$ $\quad \frac{\sin}{\cos}\cdot\cos$

$= \hat{i} - \sin t\,\hat{j}$

(ii)

$\mathbf{N} = \dfrac{\frac{d\mathbf{T}}{dt}}{\left|\frac{d\mathbf{T}}{dt}\right|} = \dfrac{-\cos t\,\hat{j}}{\cos t} = -\hat{j}$

(iii) $\kappa = \dfrac{1}{\|\mathbf{r}'(t)\|} \cdot \left|\dfrac{d\mathbf{T}}{dt}\right| = 1$

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#9 $\mathbf{r}(t) = 3\sin t\,\hat{i} + 3\cos t\,\hat{j} + 4t\,\hat{k}$

(i) $\mathbf{T} = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \dfrac{3\cos t\,\hat{i} - 3\sin t\,\hat{j} + 4\hat{k}}{\underbrace{\sqrt{9\cos^2 t + 9\sin^2 t + 16}}_{5}}$

$\mathbf{T} = \frac{3}{5}\cos t\,\hat{i} - \frac{3}{5}\sin t\,\hat{j} + \frac{4}{5}\hat{k}$

ii $\mathbf{N} = \dfrac{\frac{d\mathbf{T}}{dt}}{\left|\frac{d\mathbf{T}}{dt}\right|} = \dfrac{-\frac{3}{5}\sin t\,\hat{i} - \frac{3}{5}\cos t\,\hat{j} + 0}{3/5}$

$= -\sin t\,\hat{i} - \cos t\,\hat{j}$ ✓

#13 $\mathbf{r}(t) = \frac{t^3}{3}\hat{i} + \frac{t^2}{2}\hat{j} \qquad t > 0$

$\mathbf{T} = \dfrac{t^2\hat{i} + t\hat{j}}{\sqrt{t^4 + t^2}} = \dfrac{t^2\,\hat{i}}{t\sqrt{t^2+1}} + \dfrac{t\,\hat{j}}{t\sqrt{t^2+1}}$

$$T = \frac{t}{\sqrt{t^2+1}}\hat{i} + \frac{1}{\sqrt{t^2+1}}\hat{j}$$

$$N =$$

$$= t\left(t^2+1\right)^{-1/2}\hat{i} + \left(t^2+1\right)^{-1/2}\hat{j}$$

$$= \left[\left(t^2+1\right)^{-1/2} + t\left(-\frac{2t}{2\sqrt{t^2+1}}\right)\right]\hat{i} + \frac{-t^2}{\sqrt{t^2+1}}\hat{j}$$

unitize