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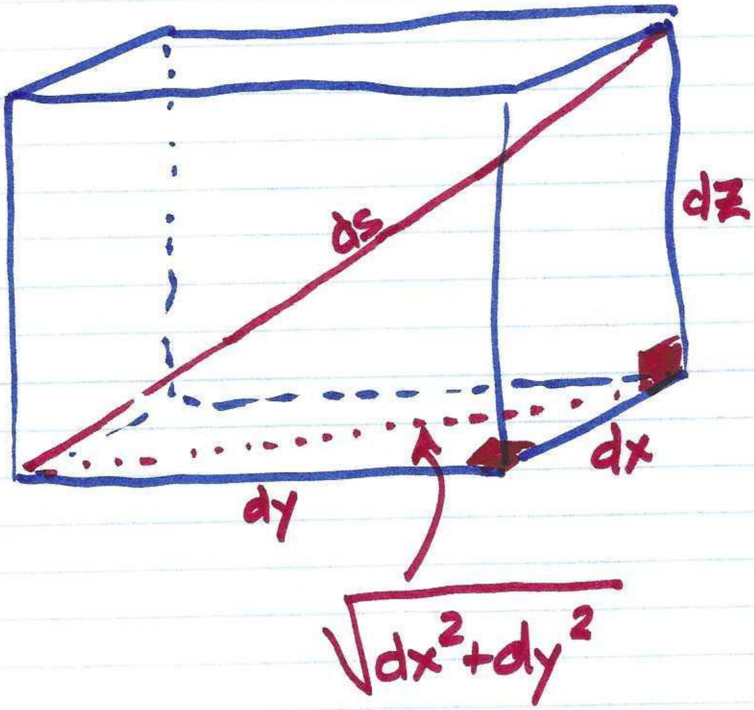
Curve Length (Rectification of Space Curves)



$$\Rightarrow ds = \sqrt{dx^2+dy^2+dz^2}$$

$$L = \int_{\text{start}}^{\text{stop}} \underbrace{\sqrt{\frac{dx(t)^2}{dt^2}+\frac{dy(t)^2}{dt^2}+\frac{dz(t)^2}{dt^2}}}_{\text{simplify how ?}}\, dt$$

$$r(t) = x(t)\,\hat{i} + y(t)\,\hat{j} + z(t)\,\hat{k}$$

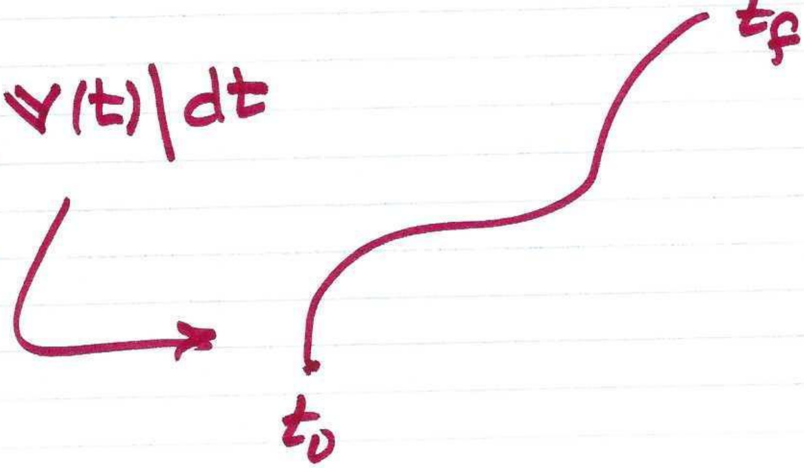
$$r'(t) = v(t) = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}$$

$$\mathbf{v}(t)\cdot\mathbf{v}(t) = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2$$

$$\sqrt{\mathbf{v}(t)\cdot\mathbf{v}(t)}\,dt = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\cdot dt$$

$$L = \int_{t_0}^{t_f} \left(\mathbf{v}(t)\cdot\mathbf{v}(t)\right)^{1/2} dt$$

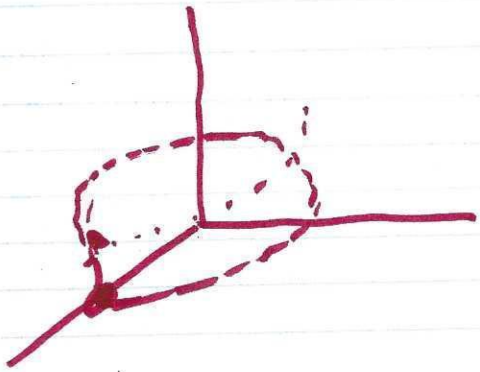
$$= \int_{t_0}^{t_f} |\mathbf{v}(t)|\, dt$$



Ex:

$$\mathbf{r}(t) = \cos t\,\hat{i} + \sin t\,\hat{j} + t\,\hat{k}$$

$$t \in [0, 2\pi]$$



$$L = \int_{t=0}^{2\pi} \sqrt{2}\,dt = \Big[\sqrt{2}\,t\Big]_0^{2\pi} = 2\sqrt{2}\pi \qquad 2\pi$$

circle only

$$\mathbf{r}'(t) = -\sin t\,\hat{i} + \cos t\,\hat{j} + (1)\,\hat{k}$$

$$\mathbf{v}(t) = |\mathbf{r}'(t)| = \sqrt{\sin^2 t + \cos^2 t + 1}$$

$$T(t) = \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|}$$

Ex. Find unit tangent where

$$\mathbf{r}(t) = (1 + 3\cos t)\,\hat{i} + (3\sin t)\,\hat{j} + (t^2)\,\hat{k}$$

$$\mathbf{r}'(t) = (-3\sin t)\,\hat{i} + (3\cos t)\,\hat{j} + (2t)\,\hat{k}$$

$$\|\mathbf{r}'(t)\| = \sqrt{9\sin^2 t + 9\cos^2 t + 4t^2}$$

$$= \sqrt{9 + 4t^2}$$

$$T(t) = \frac{1}{\sqrt{9+4t^2}}\left[(-3\sin t)\hat{i} + (3\cos t)\hat{j} + (2t)\hat{k}\right]$$

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ds is differential arc length

Recall $s(t)$ is distance function from some assumed starting point.

Frenêt co-ord.

We want $\dfrac{dr}{ds}$

Note $\dfrac{ds}{dt} \neq 0 \Rightarrow \dfrac{1}{ds/dt} = \dfrac{dt}{ds}$ exists

$$\frac{dr}{ds} = \frac{dr}{dt}\cdot\frac{dt}{ds} = v(t)\cdot\frac{1}{|v(t)|} = T(t)$$

1 $\mathbf{r}(t) = (2\cos t)\hat{i} + (2\sin t)\hat{j} + (\sqrt{5}\,t)\hat{k}$

$t \in [0, \pi]$

$\mathbf{r}'(t) = (-2\sin t)\hat{i} + (2\cos t)\hat{j} + (\sqrt{5})\hat{k}$

$|\mathbf{v}(t)| = \sqrt{\mathbf{r}'(t)\cdot\mathbf{r}'(t)}$

$= \sqrt{4\sin^2 t + 4\cos^2 t + 5}$

$= 3$

$\mathbf{r}'(t)$!

$\mathbf{T}(t) = \frac{1}{3}\left[\begin{array}{c}(-2\sin t)\hat{i} + (2\cos t)\hat{j} + \sqrt{5}\hat{k} \\ \cancel{(2\cos t)\hat{i} + (2\sin t)\hat{j} + (\sqrt{5}t)\hat{k}}\end{array}\right]$

$s(\pi) = \int_0^{\pi} 3\,dt = 3\pi$

3 $\mathbf{r}(t) = t\hat{i} + \left(\frac{2}{3}t^{3/2}\right)\hat{k} \qquad 0 \le t \le 8$

$\mathbf{r}'(t) = \hat{i} + \sqrt{t}\,\hat{k}$

$|\mathbf{r}'(t)| = \sqrt{1+t}$

3 cont'd

$$L = \int_0^8 \sqrt{1+t}\,dt \overset{u=1+t}{=} \int_{u=1}^{9} u^{1/2}du = \left[\frac{2}{3}u^{3/2}\right]_1^9 =$$

$$\frac{2}{3}(27-1) = \frac{52}{3}$$

NB:

$$\mathbb{T} = \frac{1}{\sqrt{1+t}}\left[\hat{\imath} + \sqrt{t}\,\hat{k}\right]$$

5 $\mathbb{r}(t) = \cos^3 t\,\hat{\jmath} + \sin^3 t\,\hat{k}$

$$\mathbb{r}'(t) = 3\cos^2 t(-\sin t)\hat{\jmath} + 3\sin^2 t\cos t\cdot\hat{k}$$

$$= 3\frac{\sin 2t}{2}\left[-\cos t\,\hat{\jmath} + \sin t\,\hat{k}\right].$$

$$|\mathbb{r}'(t)| = \sqrt{9\cos^4 t\sin^2 t + 9\sin^4 t\cos^2 t}$$

Aside

$$9\cos^2 t\left[\cos^2 t\sin^2 t\right] + 9\sin^2 t\left[\cos^2 t\sin^2 t\right]$$

$$= \cos^2 t\sin^2 t\left(9\cos^2 t + 9\sin^2 t\right)$$

$$= 9\left(\sin^2 t\cos^2 t\right)$$

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So... $|\mathbf{r}'(t)| = \sqrt{9\sin^2 t\cos^2 t}$

$= 3\sin t\cos t = \frac{3}{2}\sin 2t$

$$\mathbf{T}(t) = \frac{\frac{3}{2}\sin 2t\left[-\cos t\,\hat{j} + \sin t\,\hat{k}\right]}{\frac{3}{2}\sin 2t}$$

$= -\cos t\,\hat{j} + \sin t\,\hat{k}$

Find length for $0 \le t \le \pi/2$

$$L = \int_0^{\pi/2}\left(\frac{3}{2}\sin 2t\right)dt = \frac{3}{2}\left[-\frac{\cos 2t}{2}\right]_0^{\pi/2} =$$

$$= \frac{3}{4}\left[1+1\right] = \frac{3}{2}$$

8. $r(t) = (t\sin t + \cos t)\hat{i} + (t\cos t - \sin t)\hat{j}$

$+\sqrt{2} \le t \le 2$

$r'(t) = (\cancel{\sin t} + t\cos t - \cancel{\sin t})\hat{i} +$

$(\cancel{\cos t} + (-t\sin t) - \cancel{\cos t})\hat{j}$

$r'(t) = t\cos t\,\hat{i} - t\sin t\,\hat{j}$

$|r'(t)| = \sqrt{t^2\cos^2 t + t^2\sin^2 t} =$

$= t$

So $T = \cos t\,\hat{i} - \sin t\,\hat{j}$

$L = \int_{\sqrt{2}}^{2} t\,dt = \left[\frac{t^2}{2}\right]_{\sqrt{2}}^{2} = 2 - 1 = 1$

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$$\mathbf{r}(t) = (\sqrt{2}t)\hat{i} + (\sqrt{2}t)\hat{j} + (1-t^2)\hat{k}$$

arclength from $(0,0,1)$ to $(\sqrt{2},\sqrt{2},0)$

$$(0,0,1) \downarrow t=0 \qquad (\sqrt{2},\sqrt{2},0) \downarrow t=1$$

$$L = \int_0^1 |\mathbf{r}'(t)|\,dt$$

$$\mathbf{r}'(t) = \sqrt{2}\,\hat{i} + \sqrt{2}\,\hat{j} - 2t\,\hat{k}$$

$$|\mathbf{r}'(t)| = \sqrt{2+2+4t^2} = 2\sqrt{1+t^2}$$

$$L = 2\int_0^1 \sqrt{1+t^2}\,dt$$

$$\cancel{\begin{aligned} u &= 1+t^2 \\ du &= 2t\,dt \end{aligned}}$$

$$L = 2\int_{\theta=0}^{\pi/4} \sec\theta \sec^2\theta\,d\theta$$

$$t = \tan\theta$$

$$1+t^2 = \sec^2\theta$$

$$dt = \sec^2\theta\,d\theta$$

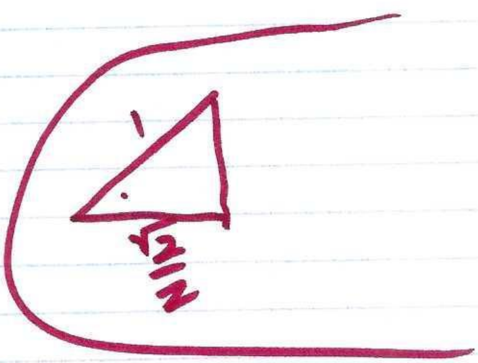
(10)

$$L = 2\int_0^{\pi/4} \sec^3\theta \, d\theta = \frac{\sec\theta \tan\theta}{2} +$$

$$\frac{1}{2}\int \sec\theta \, d\theta \to \ln|\sec\theta + \tan\theta|$$

$$L = 2\left[\frac{\sec\theta \tan\theta}{2} + \frac{1}{2}\ln|\sec\theta + \tan\theta|\right]_0^{\pi/4}$$

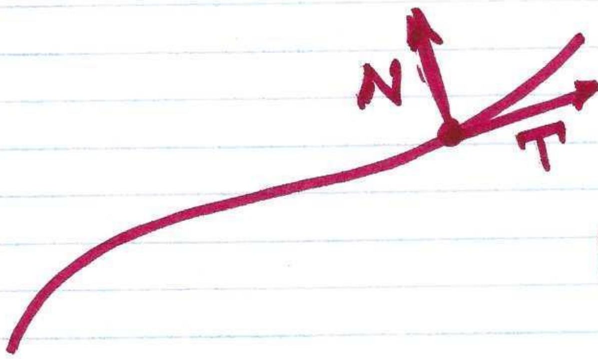
$$= 2\left[\left(\frac{\sqrt{2}(1)}{2} + \frac{\ln|\sqrt{2}+1|}{2}\right) - (0)\right]$$



$$= \sqrt{2} + \ln(\sqrt{2}+1)$$

Preview

"moving trihedron"



$B = T \times N$

pitch
yaw
roll