9/12

13.2 Vector Integration & Ballistic Trajectories

$$\int \mathbf{r}(t)\,dt := \hat{\imath}\int x(t)\,dt + \hat{\jmath}\int y(t)\,dt + \hat{k}\int z(t)\,dt$$

Ex: $\int \left[(\cos t)\hat{\imath} + (1)\hat{\jmath} - (2t)\hat{k} \right] dt =$

$$(\sin t)\hat{\imath} + (t)\hat{\jmath} - (t^2)\hat{k} + \mathbb{C}$$

Usual rules apply to definite integrals

FTC: $\int_a^b \mathbf{r}(t)\,dt = \mathbb{R}(t)\Big|_a^b$

Ex: Hang Glider Unknown path

$$\ddot{\mathbf{r}}(t) = -3\cos t\,\hat{\imath} - 3\sin t\,\hat{\jmath} + 2\hat{k}$$

$$\dot{\mathbf{r}}(0) = 3\hat{\jmath}, \quad \mathbf{r}(0) = 4\hat{\imath} + 0\hat{\jmath} + 0\hat{k}$$

②

$\dot{\mathbf{r}}^*(t)$ $\int \ddot{\mathbf{r}}(t)\,dt = -3\sin t\,\hat{i} + 3\cos t\,\hat{j} + 2t\,\hat{k}$

$3\hat{j} = \dot{\mathbf{r}}(0) = -3\underbrace{\sin 0}_{0}\,\hat{i} + 3\underbrace{\cos 0}_{1}\,\hat{j} + \underbrace{2\cdot 0}_{0}\,\hat{k} + \mathbf{C}$

$\Rightarrow \mathbf{C} = \mathbf{0}$

To get position, integrate velocity

$\mathbf{r}(t) = 3\cos t\,\hat{i} + 3\sin t\,\hat{j} + t^2\hat{k} + \mathbf{C}_1$

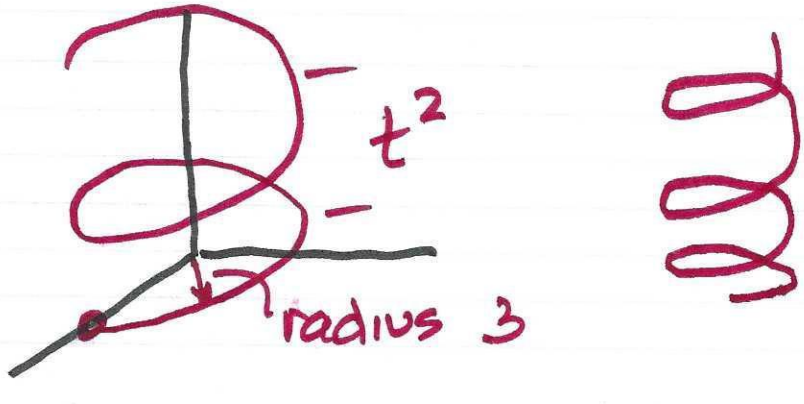
$\mathbf{r}(0) = 3\cos 0\,\hat{i} + 3\sin 0\,\hat{j} + 0^2\hat{k} + \mathbf{C}_1$

$= 4\hat{i} + 0\hat{j} + 0\hat{k}$

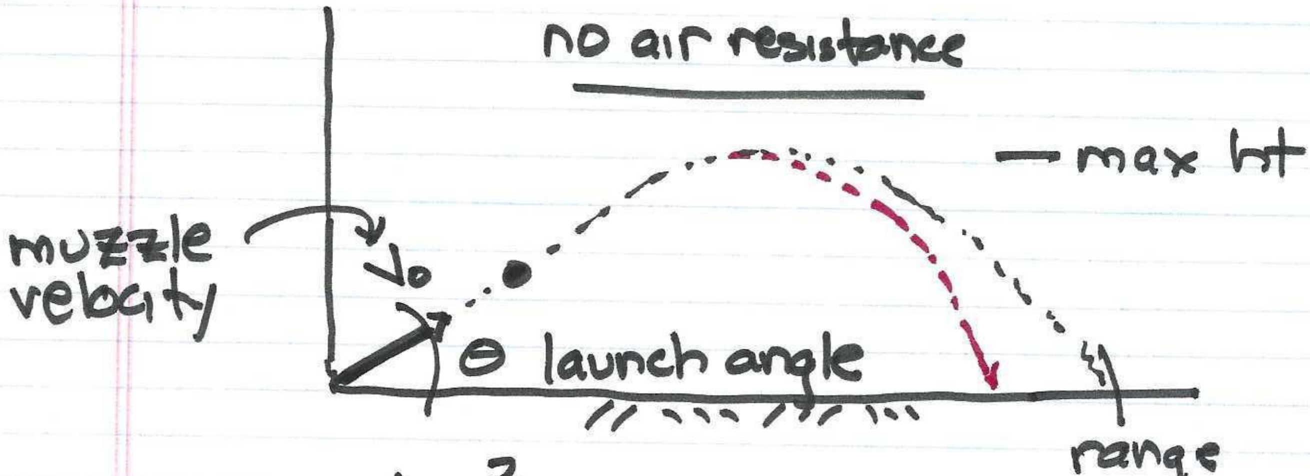
Can write $3\hat{i} + \mathbf{C}_1 = 4\hat{i}$ so $\mathbf{C}_1 = (1)\,\hat{i}$

$\mathbf{r}(t) = 3\cos t\,\hat{i} + 3\sin t\,\hat{j} + t^2\hat{k} + \hat{i}$

$= (3\cos t + 1)\,\hat{i} + 3\sin t\,\hat{j} + t^2\hat{k}$



Ballistics



$\underline{-kv^2}$

$$\mathbf{v}(t) = \cancel{\quad}$$
$$= v_0 \cos\theta \,\hat{i} + v_0 \sin\theta \,\hat{j}$$

Ballistic Eqn.

$$\mathbf{r}(t) = v_0 \cos\theta t \hat{i} + \left(v_0 \sin\theta t - \tfrac{1}{2} g t^2\right)\hat{j}$$

Ex. Given $v_0 = 500 \, m/s$, $\theta = 60°$, $g = 9.8 \, \frac{m}{s^2}$

$t = 10$ ... where is projectile?

$$\mathbf{r}(10) = (500)\left(\tfrac{1}{2}\right)(10)\,\hat{i} + \left[(500)(.866)(10 - \left(\tfrac{1}{2}\right)(9.8)(100)\right] = ?$$

$$\vec{r}(10) = 500\,\hat{i} + \underline{3840}\,\hat{j}$$

$$\boxed{\text{Max ht}} = \frac{(V_0 \sin\theta)^2}{2g} = \frac{[(500)(.866)]^2}{2 \cdot 9.8} =$$

$$\underline{9566\ \text{m}}$$

$$\boxed{\text{Flight time}} = t = \frac{2V_0 \sin\theta}{g}$$

$$\boxed{\text{Range}} = R = \frac{V_0^2}{g}\sin 2\theta$$

for example: $\dfrac{(500)^2}{9.8}(.866) = \underline{22{,}092.48\ \text{m}}$

#3 $$\int_{-\frac{\pi}{4}}^{\pi/4} \left[(\sin t)\hat{i} + (1+\cos t)\hat{j} + (\sec^2 t)\hat{k}\right] dt$$

$$= \hat{i}\int_{-\frac{\pi}{4}}^{\pi/4} \sin t\,dt + \hat{j}\int_{-\frac{\pi}{4}}^{\pi/4} (1+\cos t)\,dt + \hat{k}\int_{-\frac{\pi}{4}}^{\pi/4} \sec^2 t\,dt$$

5

$$\hat{i}(-\cos t)\Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}} + \hat{j}(t+\sin t)\Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}} + \hat{k}(\tan t)\Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$$

$$= \hat{i}\left(-\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\right)+\hat{j}\left(\frac{\pi}{4}+\frac{\sqrt{2}}{2}+\frac{\pi}{4}+\frac{\sqrt{2}}{2}\right)+\hat{k}(1+1)$$

$$= 0\,\hat{i} + \frac{\pi+2\sqrt{2}}{2}\,\hat{j} + 2\,\hat{k}$$

#7 $\displaystyle\int_0^1 \left[te^{t^2}\hat{i} + e^{-t}\hat{j} + (1)\hat{k}\right]dt =$

$\downarrow$

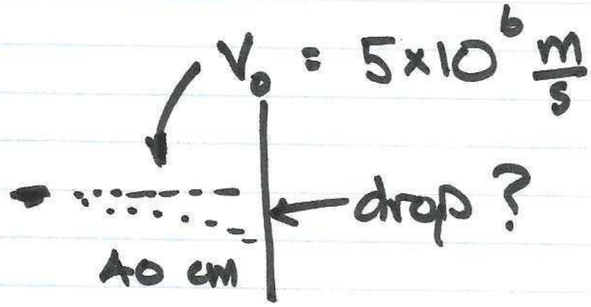
$u = t^2$

$du = 2t\,dt$

$$\hat{i}\int_{u=0}^{1} u\frac{du}{2} + \hat{j}\,-e^{-t}\Big|_0^1 + \hat{k}\,t\Big|_0^1$$

$$\hat{i}\left(\frac{1}{2}\right)+\hat{j}\left(-\frac{1}{e}+1\right)+\hat{k}(1)$$

#28



flight time $\approx \frac{0.4}{5 \times 10^6} = \frac{.08}{10^6}$ s.

8×10^{-8} s

$$\text{drop} = \frac{1}{2} g t^2 \Rightarrow \frac{9.8}{2} \cdot 64 \times 10^{-16} \text{ m}$$

$$= (9.8)(32) \times 10^{-16} \text{ m}$$

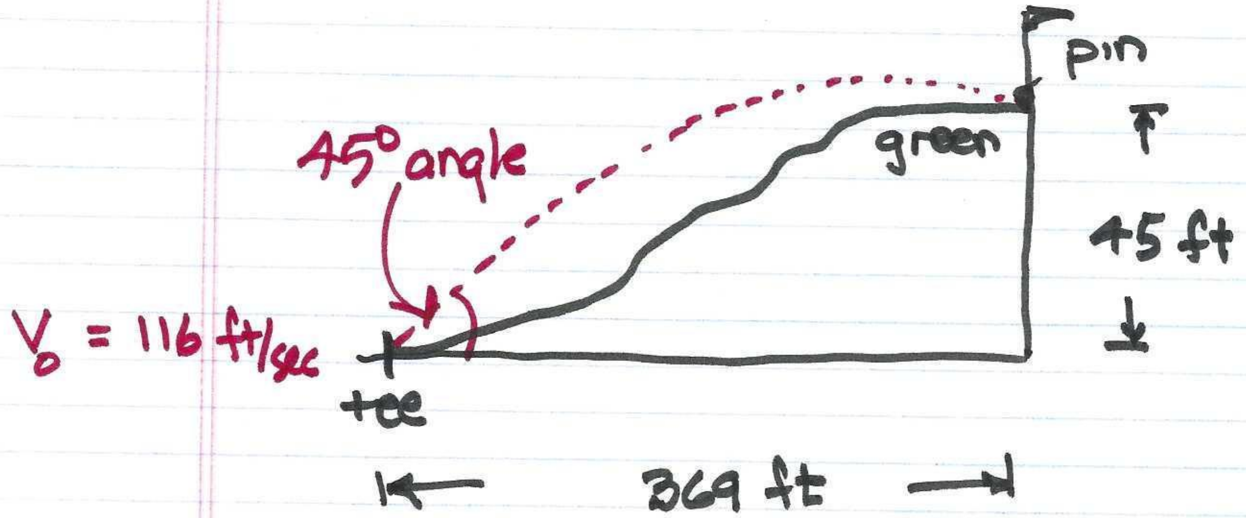
$$\cong 3.2 \times 10^{-14} \text{ m} = 3.2 \times 10^{-5} \text{ nm}$$

#30 Gun has max range 24.5 km
What is V_0

$$R = \frac{V_0^2}{2g} \cdot \sin 2\theta$$

$$24500 \text{ m} = \frac{V_0^2}{2 \cdot 9.8}$$

$$V_0^2 = 480{,}000 \Rightarrow V_0 = \boxed{692 \text{ m/s}}$$



$$\vec{r}(t) = (116)\left(\frac{\sqrt{2}}{2}\right)t\,\hat{i} + \left[(116)\left(\frac{\sqrt{2}}{2}\right)t - 16t^2\right]\hat{j}$$

$$58\sqrt{2}\,t = 369$$

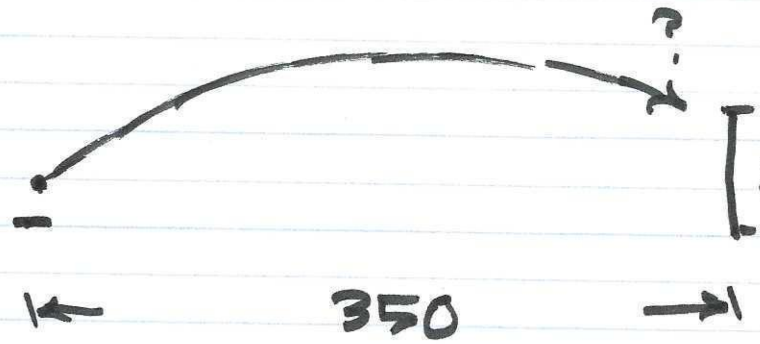
$$t = 4.5 \text{ } \underline{\text{sec}}$$

vertical position @ $t = 4.5$ is ?

$$(116)\left(\frac{\sqrt{2}}{2}\right)(4.5) - 16(4.5)^2 = \underline{45.1}$$

60 mph = 88 fps

Batter hits baseball with $V_0 = 115$ mph @ angle 25° from horizontal. Ball starts 4' above home plate. Will it clear a fence of height 20' that is 350 ft away?



So need vert position when horiz position = 350 ft. Move datum plane to actual start of flight ⇒ needs to clear 16' obstacle

Note: $V_0 = 168$ fps

$$r(t) = (168)(\cos 25^\circ)t\,\hat{i} + \left[(168)(\sin 25^\circ)t - \frac{1}{2}(32.16)t^2\right]$$

time to fence : $(168)(\cos 25^\circ)\underset{?}{t} = 350$

t_{fence} is $\underline{2.29}$

Eval $(168)(0.42)(2.29) - 16(2.29)^2 = \underline{77 \text{ ft}}$