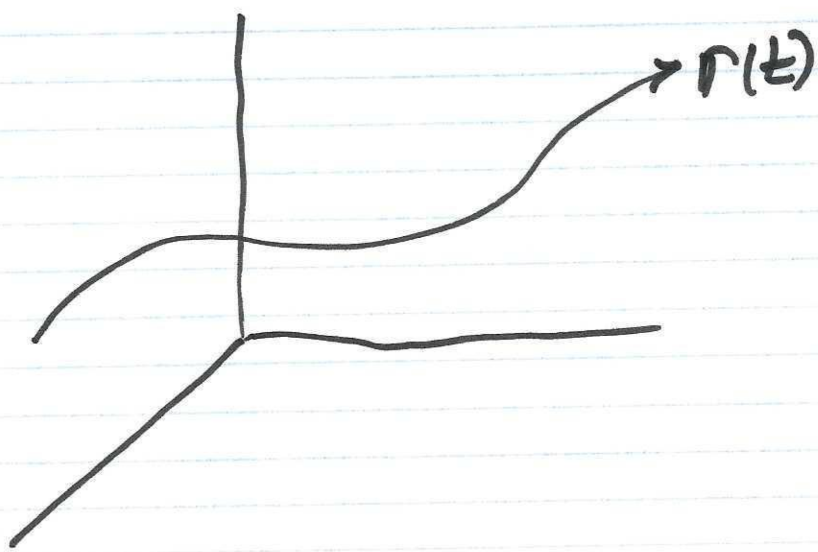


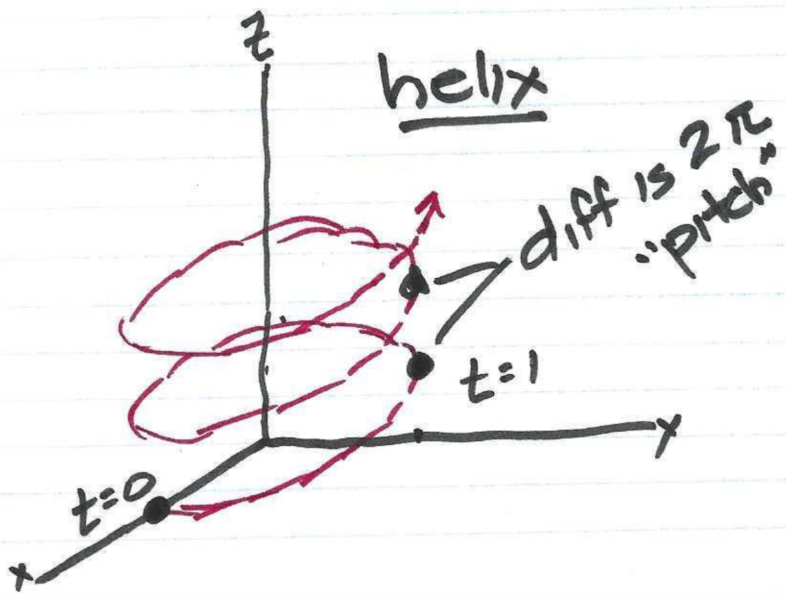
13.1 Space Curves

We define vector-valued functions as real functions attached to basis vectors

$$\mathbf{r}(t) = x(t)\,\hat{\imath} + y(t)\,\hat{\jmath} + z(t)\,\hat{k}$$



Typical: $\mathbf{r}(t) = \cos t\,\hat{\imath} + \sin t\,\hat{\jmath} + t\,\hat{k}$



Concept of limit in $\mathbb{R}^3$

$$\lim_{t \to t_0} \mathbf{r}(t) = L$$

$$\lim_{t \to t_0} \mathbf{r}(t) = \lim_{t \to t_0} x(t)\,\hat{i} + \lim_{t \to t_0} y(t)\,\hat{j} + \lim_{t \to t_0} z(t)\,\hat{k}$$

Technically $\mathbf{r}(t)$ approaches limit L as t approaches t_0 iff

Given $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$ such that whenever $0 < |t - t_0| < \delta(\varepsilon)$ that implies $|\mathbf{r}(t) - L| < \varepsilon$.

$$f(x) = \frac{x^2 - 4}{x - 2} \sim x + 2$$

$f(2)$ DNE

$f(x)$ is continuous @ $x = x_0$ iff:

1. $\lim_{x \to x_0} f(x) = L$ exists
2. $f(x_0)$ is defined
3. $f(x_0) = L$ (1 & 2 agree)

For $\mathbb{R}^3$, extend this def^{n} to the component functions.

Derivatives in $\mathbb{R}^3$

Δx is always an increment

$$\frac{d}{dt}\left(\mathbf{r}(t)\right) = \lim_{\Delta t \to 0}\left(\frac{\mathbf{r}(t+\Delta t) - \mathbf{r}(t)}{\Delta t}\right) = \mathbf{r}'(t)$$

$\mathbf{r}'(t)$ represents 3-dim velocity

$\mathbf{r}''(t)$ " " acceleration

$|\mathbf{r}'(t)|$ " " speed

Given $\mathbf{r}(t) = x(t)\,\hat{\imath} + y(t)\,\hat{\jmath} + z(t)\,\hat{k}$

$$\mathbf{r}'(t) = x'(t)\,\hat{\imath} + y'(t)\,\hat{\jmath} + z'(t)\,\hat{k}$$

The vector $\mathbf{r}'(t)$ is <u>tangent</u> to $\mathbf{r}(t)$

So... the unit tangent vector $T(t)$ is :

$$T(t) := \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

Example:

$$\mathbf{r}(t) = 2\cos t\,\hat{\imath} + 2\sin t\,\hat{\jmath} + 5\cos^2 t\,\hat{k}$$

velocity is $\mathbf{r}'(t) = -2\sin t\,\hat{\imath} + 2\cos t\,\hat{\jmath} \overset{(-5\sin 2t)}{- 10\sin t\cos t}\,\hat{k}$

acceleration is $\mathbf{r}''(t) = -2\cos t\,\hat{\imath} - 2\sin t\,\hat{\jmath} - 10\cos 2t\,\hat{k}$

unit tangent is $T(t) = \dfrac{-2\sin t\,\hat{\imath} + 2\cos t\,\hat{\jmath} - 5\sin 2t\,\hat{k}}{\sqrt{4 + 25\sin^2 2t}}$

Differentiation Rules for Vector Functions:

1) Constant function $\frac{d}{dt}(C) = 0$

2) Scalar multiples:

(i) $\frac{d}{dt}\left[c\,\mathbf{r}(t)\right] = c\,\mathbf{r}'(t)$

(ii) $\frac{d}{dt}\left[f(t)\,\mathbf{r}(t)\right] = f'(t)\,\mathbf{r}(t) + f(t)\,\mathbf{r}'(t)$

3) Sum/Difference

$$\frac{d}{dt}\left[\mathbf{r}(t) \pm \mathbf{s}(t)\right] = \mathbf{r}'(t) \pm \mathbf{s}'(t)$$

4) Dot Product

$$\frac{d}{dt}\left[\mathbf{r}(t)\cdot\mathbf{s}(t)\right] = \mathbf{r}'(t)\cdot\mathbf{s}(t) + \mathbf{r}(t)\cdot\mathbf{s}'(t)$$

5) Cross Product

$$\frac{d}{dt}\left[\mathbf{r}(t)\times\mathbf{s}(t)\right] = \mathbf{r}'(t)\times\mathbf{s}(t) + \mathbf{r}(t)\times\mathbf{s}'(t)$$

6) Chain Rule

$$\frac{d}{dt}\left[\mathbf{r}(f(t))\right] = f'(t)\cdot\mathbf{r}'(f(t))$$

Thm If $\mathbf{r}(t)$ is differentiable and $|\mathbf{r}(t)|$ is constant, then $\mathbf{r}\cdot\frac{d\mathbf{r}}{dt} = 0$

Pf: $\mathbf{r}(t)\cdot\mathbf{r}(t) = |\mathbf{r}(t)|^2 = \text{const}$

diff: $\mathbf{r}'(t)\cdot\mathbf{r}(t) + \mathbf{r}(t)\cdot\mathbf{r}'(t) = 0$

$= 2\,\mathbf{r}'(t)\cdot\mathbf{r}(t) = 0$

$\Rightarrow \mathbf{r}(t)\cdot\mathbf{r}'(t) = 0$ $\boxtimes$

Exercises:

1. $$\lim_{t\to\pi}\left[\sin\frac{t}{2}\,\hat{i}+\cos\frac{2}{3}t\,\hat{j}+\tan\frac{5}{4}t\,\hat{k}\right]$$

$$=\sin\frac{\pi}{2}\,\hat{i}+\cos\frac{2\pi}{3}\,\hat{j}+\tan\frac{5\pi}{4}\,\hat{k}$$

$$=(1)\,\hat{i}-\left(\tfrac{1}{2}\right)\hat{j}+(1)\,\hat{k}$$

9. $x^2+y^2=1$

$\mathbf{r}(t)=\sin t\,\hat{i}+\cos t\,\hat{j}$ @ $\frac{\pi}{4}$

@ $t=\frac{\pi}{4}$

vel: $\mathbf{r}'(t)=\cos t\,\hat{i}-\sin t\,\hat{j}$ | $\frac{\sqrt{2}}{2}(\hat{i}-\hat{j})$

acc: $\mathbf{r}''(t)=-\sin t\,\hat{i}-\cos t\,\hat{j}$ | $\frac{\sqrt{2}}{2}(-\hat{i}-\hat{j})$

13. $\mathbf{r}(t)=(t+1)\,\hat{i}+(t^2-1)\,\hat{j}+2t\,\hat{k}$ @ $\underline{t=1}$

$\mathbf{r}'(t)=\hat{i}+2t\,\hat{j}+2\hat{k}$

$\mathbf{r}'(1)=\hat{i}+2\hat{j}+2\hat{k}$ $\quad$ $\mathbf{r}''(t)=2\hat{j}$

$|\mathbf{r}'(1)|=3$

$T(1)=\frac{1}{3}(\hat{i}+2\hat{j}+2\hat{k})$

(23) $\mathbf{r}(t) = \sin t\,\hat{\imath} + (t^2 - \cos t)\,\hat{\jmath} + e^t\,\hat{k} \;\Big|\; t_0 = 0$

$\mathbf{r}'(t) = \cos t\,\hat{\imath} + (2t + \sin t)\,\hat{\jmath} + e^t\,\hat{k}$

$\mathbf{r}'(0) = \hat{\imath} + 2\hat{\jmath} + \hat{k}$

$\underline{\mathbf{r}_0 + \mathbf{v}t}$

$\mathbf{r}(0) = 0\hat{\imath} + (-1)\hat{\jmath} + (1)\hat{k} = -\hat{\jmath} + \hat{k}$ is point on line

$$\mathbf{v}(t) = \underbrace{\left(-\hat{\jmath} + \hat{k}\right)}_{\mathbf{r}_0} + \overset{t\,\cdot\,\mathbf{v}}{t\left(\hat{\imath} + 2\hat{\jmath} + \hat{k}\right)}$$

$(0, -1, 1)$

point of tangency

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(47) If $\mathbf{r}(t)$ is differentiable @ t_0, then it is continuous @ t_0.

Pf: $\displaystyle\lim_{t \to t_0} \left(\frac{\mathbf{r}(t) - \mathbf{r}(t_0)}{t - t_0} \right) = \mathbf{r}'(t_0)$

So $\mathbf{r}(t) - \mathbf{r}(t_0) = \mathbf{r}'(t_0) \cdot (t - t_0)$

Want $\displaystyle\lim_{t \to t_0} \mathbf{r}(t)$ to be $\mathbf{r}(t_0)$

47 cont'd

Note given $\varepsilon > 0$ we can find $\delta(\varepsilon) > 0 \ni$

$$\varepsilon > |\mathbf{r}(t) - \mathbf{r}(t_0)| = \underline{|\mathbf{r}'(t_0)|\,(t - t_0)}$$

Choose $|\mathbf{r}'(t_0)|(t - t_0) < \varepsilon$

or $t - t_0 < \frac{\varepsilon}{|\mathbf{r}'(t_0)|} = \delta(\varepsilon)$ ∎