8/27

$\mathbf{v} = \langle v_1, v_2, v_3 \rangle = v_1 \hat{i} + v_2 \hat{j} + v_3 \hat{k}$

Dot Product (or Inner Product)

Given $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$, $\mathbf{w} = \langle w_1, w_2, w_3 \rangle$

$$\mathbf{v} \cdot \mathbf{w} := v_1 w_1 + v_2 w_2 + v_3 w_3$$

higher dimensions $\langle a_i \rangle, \langle b_i \rangle$

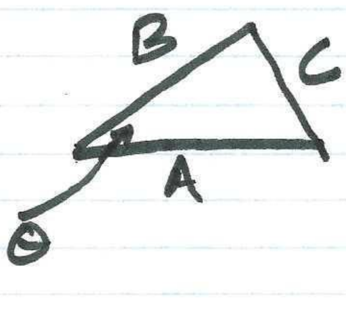
$$A \cdot B = \sum_{i=1}^{n} a_i b_i$$

Physics Version

$$\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}||\mathbf{w}| \cos \theta$$

Why are these the same?

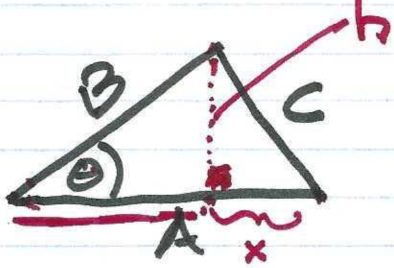
Use LOC - Law of Cosines



LOC says:

$$C^2 = A^2 + B^2 - 2AB \cos \theta$$

Proof of Law of Cosines



$$C^2 = h^2 + x^2$$

$$(A - x)^2 + h^2 = B^2$$

$$A^2 - 2Ax + \underbrace{x^2 + h^2} = B^2$$

$$A^2 - 2Ax + C^2 = B^2$$

$$A - x = B\cos\theta$$

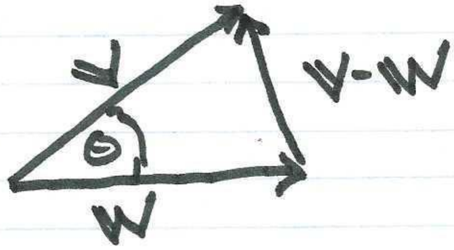
square $\quad x = A - B\cos\theta$

$$x^2 = A^2 - 2AB\cos\theta + B^2\cos^2\theta$$

$$C^2 = h^2 + A^2 - 2AB\cos\theta + B^2\cos^2\theta$$

$$C^2 = (B\sin\theta)^2 + A^2 - 2AB\cos\theta + B^2\cos^2\theta$$

Finally $\quad C^2 = A^2 + B^2 - 2AB\cos\theta$



$$|v-w|^2 = |v|^2 + |w|^2 - 2|v||w|\cos\theta$$

$$v = \langle v_1, v_2, v_3 \rangle, \quad w = \langle w_1, w_2, w_3 \rangle$$

$$(v_1-w_1)^2 + (v_2-w_2)^2 + (v_3-w_3)^2 =$$

$$v_1^2 + v_2^2 + v_3^2 + w_1^2 + w_2^2 + w_3^2 -$$

$$2|v||w|\cos\theta$$

$$(\cancel{v_1^2} - 2v_1w_1 + \cancel{w_1^2}) + (\cancel{v_2^2} - 2v_2w_2 + \cancel{w_2^2}) +$$

$$(\cancel{v_3^2} - 2v_3w_3 + \cancel{w_3^2}) = \cancel{v_1^2} + \cancel{v_2^2} + \cancel{v_3^2} +$$

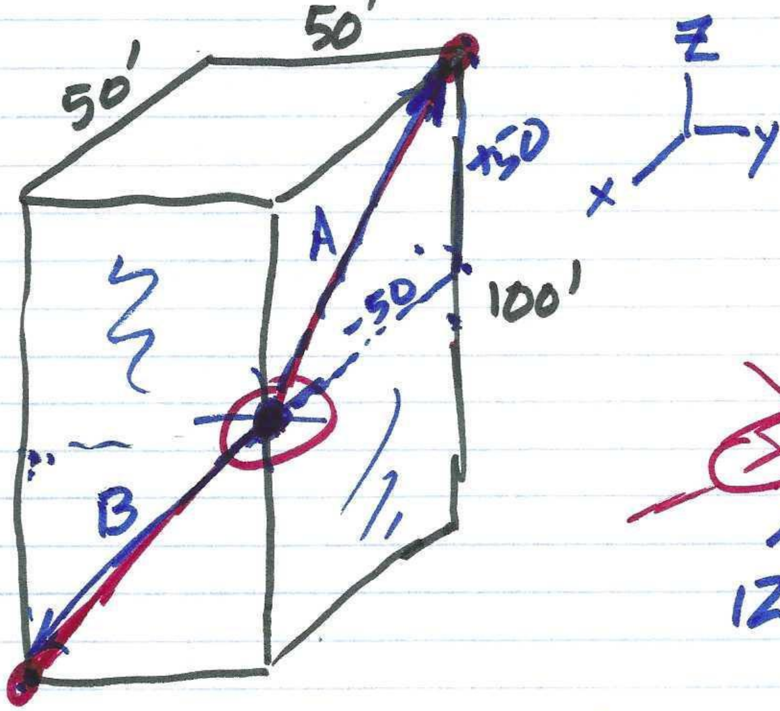
$$\cancel{w_1^2} + \cancel{w_2^2} + \cancel{w_3^2} - 2|v||w|\cos\theta$$

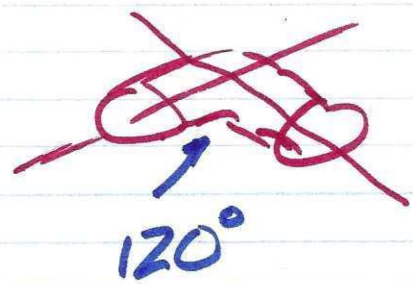
$$-2v_1w_1 - 2v_2w_2 - 2v_3w_3 = -2|v||w|\cos\theta$$

$\div -2$

$$v_1w_1 + v_2w_2 + v_3w_3 = |v||w|\cos\theta$$

4





$A = \langle -50, 0, 50 \rangle$ ✓

$B = \langle 0, -50, -50 \rangle$

!!

$A \cdot B = -2500$

$= |A|\,|B| \cos\theta$

$$\boxed{\theta = \arccos\left(\frac{A \cdot B}{|A|\,|B|}\right)}$$

~~$\cos^{-1}$~~ arccos

$$\theta = \arccos\left(\frac{-2500}{\sqrt{5000}\,\sqrt{5000}}\right) = \arccos\left(\frac{-2500}{5000}\right)$$

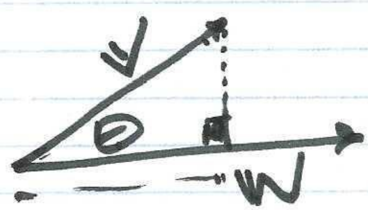
$$= \arccos\left(-\frac{1}{2}\right) = 120^\circ$$

Note: Dot product detects perpendicularity

Rules for dot product:

1. $v \cdot w = w \cdot v$
2. $v \cdot 0 = 0$
3. $u \cdot (v + w) = u \cdot v + u \cdot w$
4. $u \cdot cv = c(u \cdot v)$
5. $|u| = \sqrt{u \cdot u}$

Projections of one vector onto another

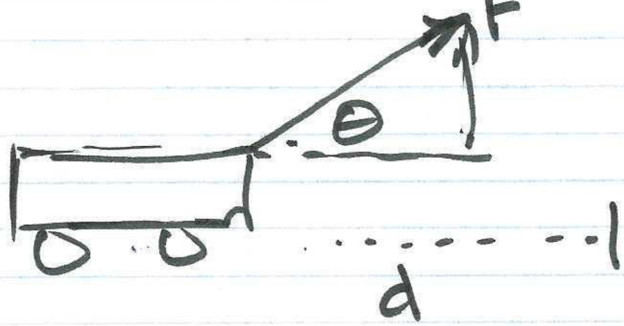


$$\mathrm{proj}_w(v) = |v| \cdot \cos\theta = \frac{\cancel{|v|}(v \cdot w)}{\cancel{|v|}\,|w|}$$

$$= \frac{v \cdot w}{|w|} = v \cdot \underbrace{\frac{w}{|w|}}_{\hat{e}_w}$$

Work

$$\underset{\text{work}}{W} = \underset{\text{force}}{F} \cdot \underset{\text{displacement}}{d}$$



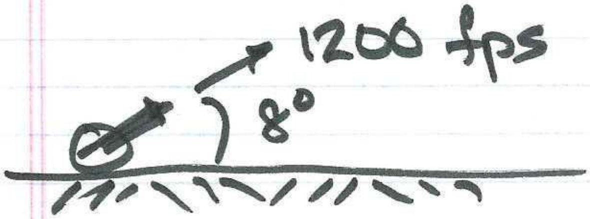
Prob's Prove Cauchy-Schwarz Inequality

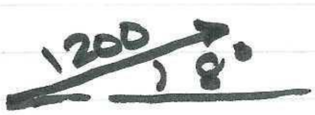
#27 p. 735

$$|u||v| \geq u \cdot v$$

$$1 \overset{?}{\geq} \frac{u \cdot v}{|u||v|} = \cos\theta$$

#25





$$V_h = 1200 \cos 8^\circ$$

$$V_v = 1200 \sin 8^\circ$$

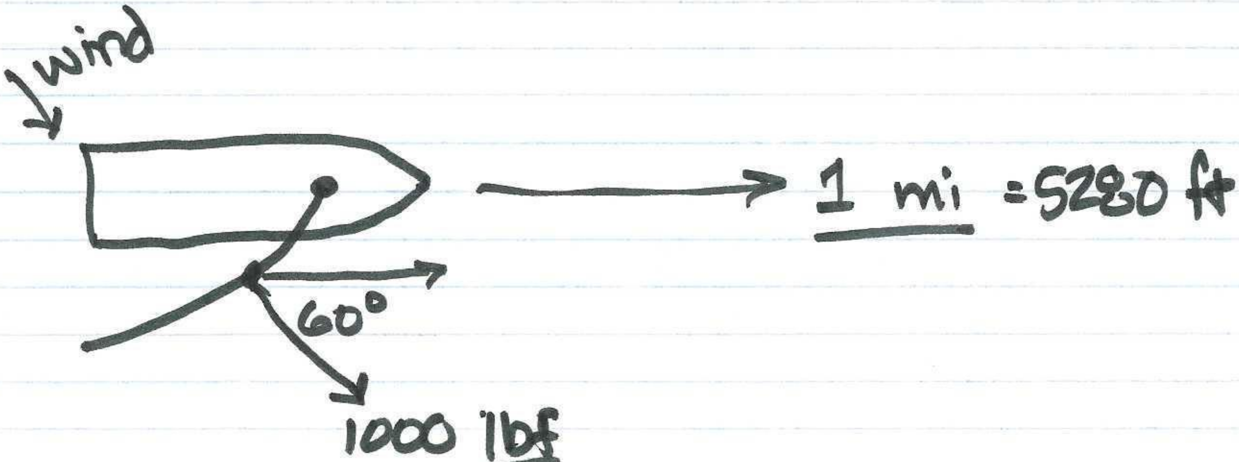
#44 p.736

$F \cdot d = ?$ $\quad 602{,}000\ N = F$

$605\ km = 605{,}000\ m = d$

$360{,}000 \cdot 10^6\ N\text{-}m = \underline{3.6 \times 10^{11}\ J}$

#46



$\underline{\text{WORK}} = (1000)(\tfrac{1}{2})(5280) = \_\_\_\_$

Cross Product:

$$\mathbb{V} = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$$

$$\mathbb{W} = w_1\hat{i} + w_2\hat{j} + w_3\hat{k}$$

parity factor

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

$$\mathbb{V} \times \mathbb{W} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} =$$

$$\hat{i}\left(v_2 w_3 - v_3 w_2\right) - \hat{j}\left(v_1 w_3 - v_3 w_1\right)$$

$$+ \hat{k}\left(v_1 w_2 - v_2 w_1\right)$$

Physics Version

$$\mathbb{V} \times \mathbb{W} = |\mathbb{V}||\mathbb{W}| \sin\theta \, \hat{e}_\perp$$

$\hat{e}_\perp$ is $\perp$ to both $\mathbb{V}$ & $\mathbb{W}$ and has direction determined by "right hand rule"

⑨

$$\mathbf{v} = 2\hat{i} + 3\hat{j} - \hat{k}$$

$$\mathbf{w} = \hat{i} - \hat{j} + 2\hat{k}$$

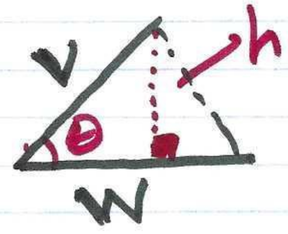
$$\mathbf{v}\times\mathbf{w} = \cancelto{0}{2\hat{i}\times\hat{i}} - \underline{2\cancel{\hat{i}\times\hat{j}}} + 2\cancel{\hat{i}\times 2\hat{k}}$$

$$+ 3\cancel{\hat{j}\times\hat{i}} - \cancelto{0}{3\hat{j}\times\hat{j}} + 3\cancel{\hat{j}\times 2\hat{k}}$$

$$- \cancel{\hat{k}\times\hat{i}} + \cancel{\hat{k}\times\hat{j}} - \cancelto{0}{\hat{k}\times 2\hat{k}}$$

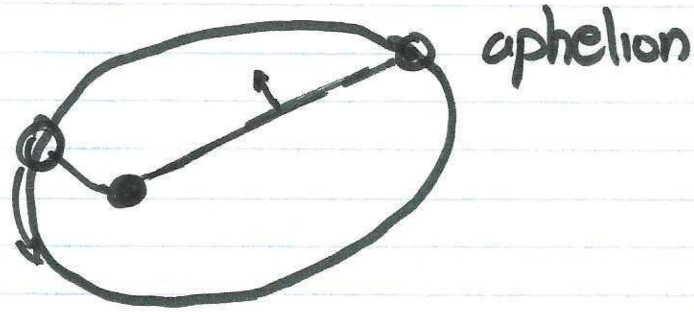
$$-2\hat{k} - 4\hat{j} - 3\hat{k} + 6\hat{i} + \hat{j} - \hat{i}$$

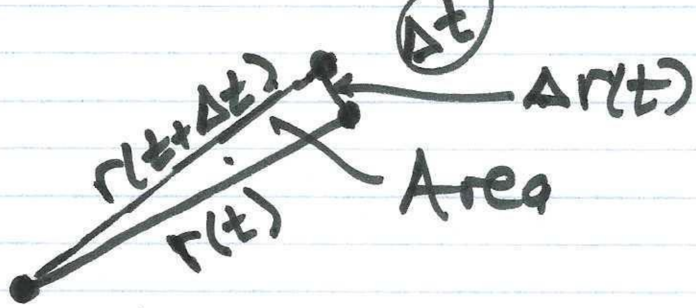
$$\mathbf{v}\times\mathbf{w} = 5\hat{i} - 3\hat{j} - 5\hat{k}$$



$$\text{Area} = \frac{1}{2}\underbrace{|\mathbf{w}||\mathbf{v}|\sin\theta}$$

$$\text{Area} = \frac{1}{2}|\mathbf{v}\times\mathbf{w}|$$





$$\Delta A = \frac{1}{2}\left| r(t) \times \Delta r(t) \right|$$

$$\frac{\Delta A}{\Delta t} = \frac{1}{2}\left| r(t) \times \frac{\Delta r(t)}{\Delta t} \right|$$

$$\lim_{\Delta t \to 0} : \frac{dA}{dt} = \frac{1}{2}\left| r(t) \times \frac{dr(t)}{dt} \right|$$

What is $\frac{d^2A}{dt^2} = \frac{1}{2}\left| \left[\underbrace{\frac{dr(t)}{dt} \times \frac{dr(t)}{dt}}_{0} + r(t) \times \frac{d^2r(t)}{dt^2} \right] \right|$

(11)

These are antiparallel

$$\frac{d^2A}{dt^2} = \frac{1}{2}\left|\dot{r}(t) \times \frac{d^2\dot{r}(t)}{dt^2}\right| = 0$$

$$F = -\frac{GM_s m_p}{r^2}\hat{e}_r$$

$$F = m_p \frac{d^2 r(t)}{dt^2}$$

So $\frac{d^2A}{dt^2} = 0 \Rightarrow \frac{dA}{dt} = \text{constant}$ this is Kepler 2