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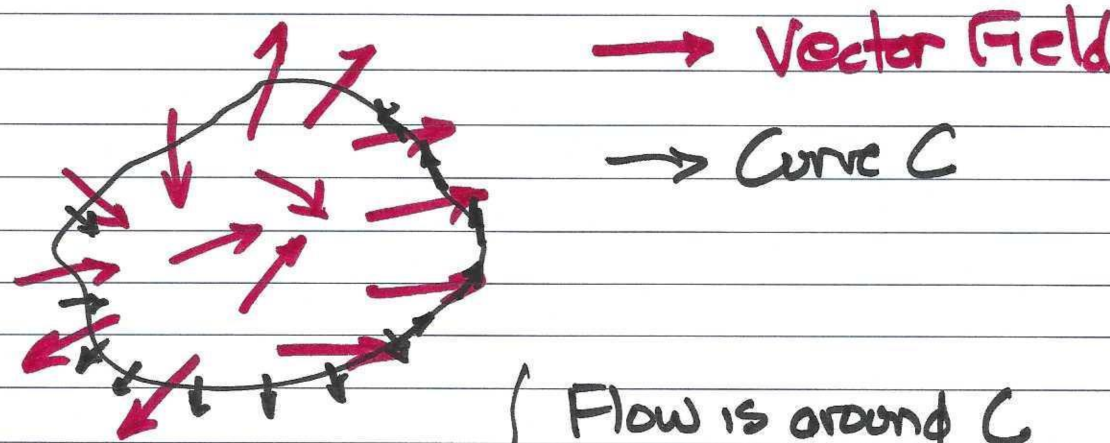
## Flux vs Flow




Flow is around C

Flux is across C

$$\text{Flow} = \oint \mathbf{F} \cdot d\mathbf{r}$$

$$\text{Flux} = \oint \mathbf{F} \cdot \hat{n}\, dt$$

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Ex. for flux integral

Flux of $\mathbf{F} = M\hat{i} + N\hat{j}$ across $C$ is:

$$\oint_C M\,dy - N\,dx$$

C is unit circle $\quad \mathbf{r}(t) = \cos t\,\hat{i} + \sin t\,\hat{j}$

$$\mathbf{F}(x,y) = (x-y)\,\hat{i} + x\,\hat{j}$$

$$\mathbf{F}(t) = \underbrace{(\cos t - \sin t)}_{M}\,\hat{i} + \underbrace{\cos t}_{N}\,\hat{j}$$

$$dx = -\sin t\,dt$$

$$dy = \cos t\,dt$$

$$\int_0^{2\pi} (\cos t - \sin t)\cos t\,dt - \int_0^{2\pi} \cos t\,(-\sin t)\,dt$$

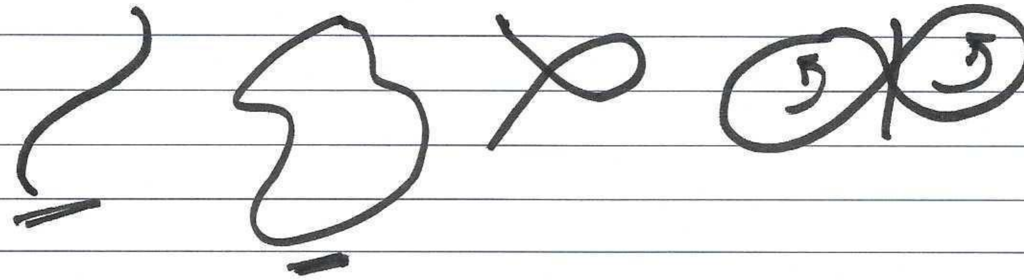
$$= \int_0^{2\pi} (\cos^2 t \underbrace{- \sin t \cos t}_{0})\,dt + \int_0^{2\pi} \underbrace{\sin t \cos t}_{0}\,dt$$

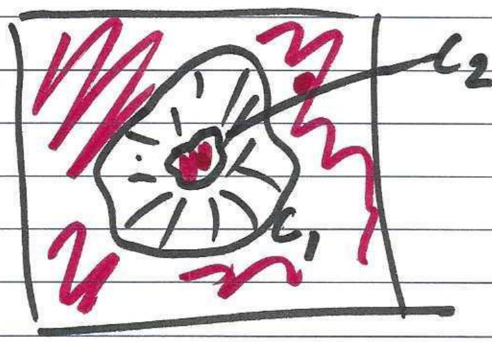
$$= \int_0^{2\pi} \cos^2 t\,dt = \int_0^{2\pi} \left(\frac{1+\cos 2t}{2}\right) dt = \left[\frac{t}{2} + \frac{\sin 2t}{4}\right]_0^{2\pi}$$

$$= \boxed{\pi}$$

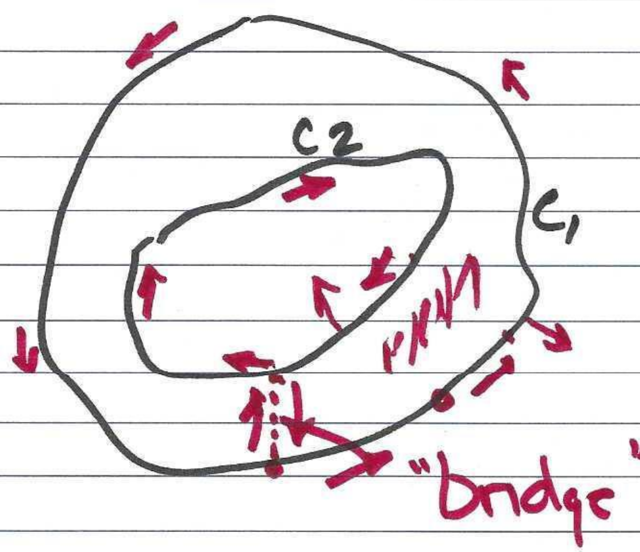
3

**Paths:**



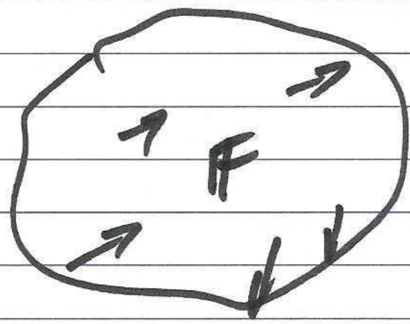



multiply connected region



# Conservative Fields

$\mathbb{F}(x,y,z)$ is a conservative vector field if and only if $\mathbb{F}(x,y,z) = \nabla f(x,y,z)$




$\nabla \times \mathbb{F} = 0$ then $\mathbb{F}$ is conservative wherever this occurs

$$\nabla \times \mathbb{F} = \nabla \times \nabla f$$

$$\mathbb{F} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\nabla \times \mathbb{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = ?$$

$$\hat{i}\left(\frac{\partial z}{\partial y}-\frac{\partial y}{\partial z}\right)-\hat{j}\left(\frac{\partial z}{\partial x}-\frac{\partial x}{\partial z}\right)+\hat{k}\left(\frac{\partial y}{\partial x}-\frac{\partial x}{\partial y}\right)$$

If these are zero, $\nabla \times \mathbb{F} = 0$

$$\nabla f = \hat{i}\frac{\partial f}{\partial x}+\hat{j}\frac{\partial f}{\partial y}+\hat{k}\frac{\partial f}{\partial z}$$

$$\nabla \times \underline{\underline{\nabla}} f = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} = ?$$

$$\hat{i}\left(\frac{\partial^2 f}{\partial y \partial z}-\frac{\partial^2 f}{\partial z \partial y}\right)-\hat{j}\left(\frac{\partial^2 f}{\partial x \partial z}-\frac{\partial^2 f}{\partial z \partial x}\right)+\hat{k}\left(\frac{\partial^2 f}{\partial x \partial y}-\frac{\partial^2 f}{\partial y \partial x}\right)$$

$$0 \qquad\qquad\qquad 0 \qquad\qquad\qquad 0$$

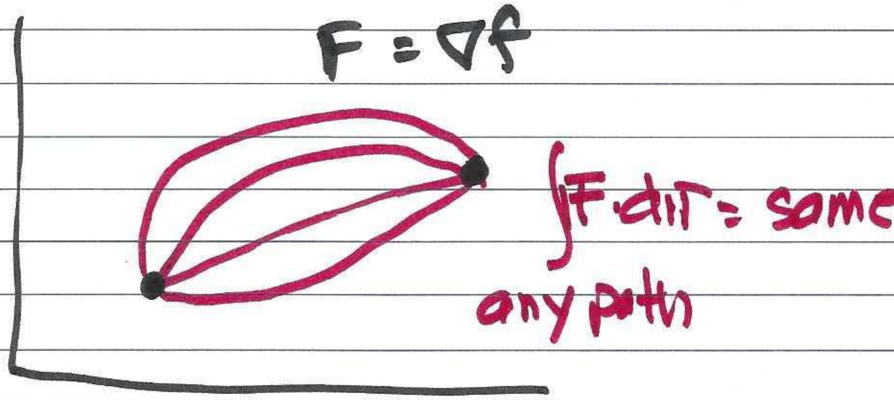
So curl is zero i.e. $\left.\begin{array}{l} \nabla \times \mathbb{F} = 0 \\ \nabla \times \nabla f = 0 \end{array}\right\}$ same

Fundamental Th$^{\text{m}}$ of Line Integrals
(says whenever integral is path independent)

reg $\int$ $\quad \int_a^b f(x)\,dx = F(b) - F(a)$

$$\mathbb{F} = \nabla f(x,y,z)$$

line $\int$ $\quad \int_C \mathbb{F}\cdot d\mathbf{r} = f(\text{end}) - f(\text{initial})$

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Ex: $\mathbb{F}(x,y,z) = \nabla\left(\dfrac{1}{x^2+y^2+z^2}\right)$

How much work (for/against) force field moving from $(\phi,0,0)$ to $(1,1,1)$

$$W = \int_C \mathbf{F} \cdot d\mathbf{r}$$

$$\mathbf{F} = \nabla f, \quad f = \frac{1}{x^2+y^2+z^2}$$

$$\text{then } \int_C \mathbf{F} \cdot d\mathbf{r} = f(1,1,1) - f(1,0,0)$$

$$= \frac{1}{3} - 1 = -\frac{2}{3}$$

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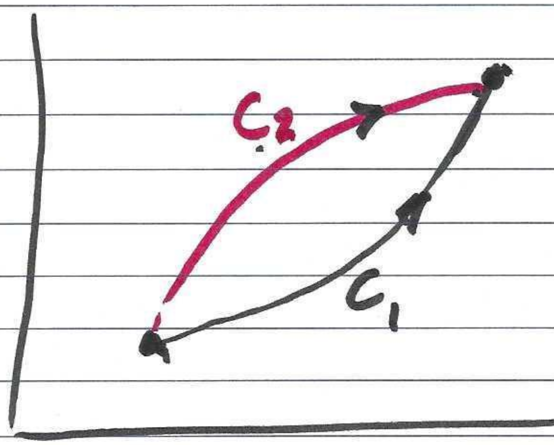
Assume $\mathbf{F} = \nabla f$

$$\int_C \mathbf{F} \cdot d\mathbf{r} \rightsquigarrow \int_A^B \nabla f \cdot d\mathbf{r} \qquad d\mathbf{r} = \mathbf{r}'(t)\,dt$$

$$\int_A^B \underline{\nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt} = \int_A^B \underline{\frac{d}{dt}\left(f(\mathbf{r}(t))\right)}\,dt$$

$$\Big[f(\mathbf{r}(t))\Big]_A^B$$




$$\int_{C_1} \mathbb{F}\cdot d\mathbf{r}$$

$$\int_{C_2} \mathbb{F}\cdot d\mathbf{r}$$

If $\mathbb{F}$ is conservative, what is the circulation? i.e.

$$\oint_{C_1 - C_2} \mathbb{F}\cdot d\mathbf{r} = 0$$

So... $$\int_{C_1} \mathbb{F}\cdot d\mathbf{r} - \int_{C_2} \mathbb{F}\cdot d\mathbf{r} = \oint_{C_1 - C_2} \mathbb{F}\cdot d\mathbf{r} = 0$$

then $\int_{C_1} \mathbb{F}\cdot d\mathbf{r} = \int_{C_2} \mathbb{F}\cdot d\mathbf{r}$ ie <u>path independent</u>

#25 $\int_A^B \overset{M}{z^2}\,dx + \overset{N}{2y}\,dy + \overset{P}{2xz}\,dz$ ← show indep of path

Vector field is conservative iff

$$\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z}, \quad \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}, \quad \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$$

$$0 \overset{\checkmark}{=} 0, \quad 2z \overset{\checkmark}{=} 2z, \quad 0 = 0$$

so **yes** conservative (per test p.994)

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#29 Find work done by $\mathbf{F} = (x^2+y^2)\,\hat{\imath} + (y^2+x)\,\hat{\jmath} + (ze^z)\,\hat{k}$

Initial point is $(1,0,0)$

Final point is $(1,0,1)$

Path 1: $x = 1, y = 0, z$ from 0 to 1

$$\mathbf{r}(t) = \hat{i} + t\hat{k} \qquad 0 \le t \le 1$$

$$\mathbf{r}'(t) = \hat{k}$$

$$\mathbf{F}(t) = \hat{i} + \hat{j} + te^t\hat{k}$$

$$\int_{C_1} (\hat{i} + \hat{j} + te^t\hat{k}) \cdot (\hat{k})\,dt =$$

$$W = \int_0^1 te^t\,dt = \left[te^t\right]_0^1 - \int_0^1 e^t\,dt$$

$$= e - e^t\Big|_0^1 = \underline{0}$$

Check: $\nabla \times F \overset{?}{=} 0$

(11)

$$\nabla \times \mathbb{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 1 & 1 & ze^{z} \end{vmatrix} =$$

$0 + 0 + 0 = 0$ STOP!