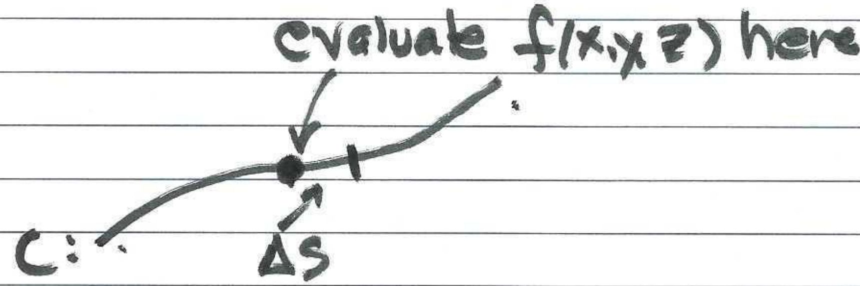
Line Integrals (continued)

Recall: fundamental definition of line integral is $\lim_{\Delta s_i \to 0} \sum_{i=1}^{n} f(x,y,z)\,\Delta s_i$



Method for Evaluation

Step 1: Parametrize the path

$$\mathbf{r}(t) = x(t)\,\hat{i} + y(t)\,\hat{j} + z(t)\,\hat{k}$$

Step 2: Re-express $f(x,y,z)$ in terms of t

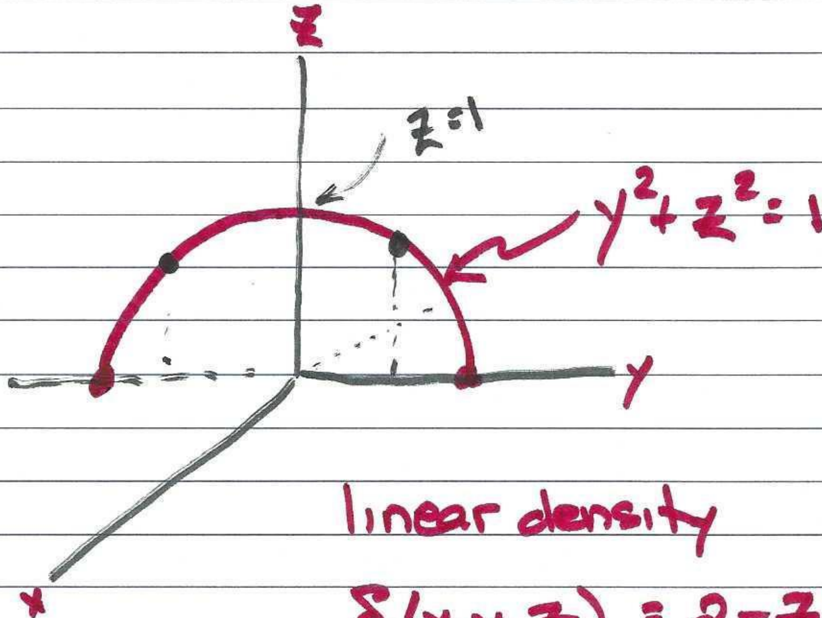
Step 3: $\int_C f\big(x(t), y(t), z(t)\big) \dfrac{ds}{dt}\,dt$

$|v(t)|$

$|r'(t)|$

Recall $ds = \dfrac{ds}{dt}\,dt$

Ex: What is mass and centroid of:



linear density

$\delta(x,y,z) = 2-z$

In gen'l: Mass: $M = \int_C \delta \, ds$ from Table 16.1

Note $\bar{x} = 0$, $\bar{y} = 0$

(1) Parametrize curve:

$$\mathbf{r}(t) = \cos t \, \hat{j} + \sin t \, \hat{k} \qquad 0 \le t \le \pi$$

2 Find $\frac{ds}{dt} = |\mathbf{r}'(t)|$

$$\mathbf{r}'(t) = -\sin t\,\hat{j} + \cos t\,\hat{k}$$

$$|\mathbf{r}'(t)| = \sqrt{\sin^2 t + \cos^2 t} = 1$$

3 Do mass integral

$$\int_0^{\pi} (2-\sin t)(1)\,dt = \Big[2t + \cos t\Big]_0^{\pi} =$$

$$(2\pi + (-1)) - (0+1) = \underline{2\pi - 2} = M$$

4 Do moment integral $\boxed{\int_C z\delta\,ds}$

$$M_{xy} = \int_0^{\pi} (\sin t)(2-\sin t)(1)\,dt$$

$$M_{xy} = \int_0^{\pi} (2\sin t - \sin^2 t)\, dt$$

$$= \frac{8-\pi}{2}$$

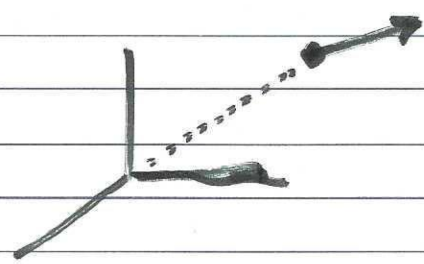
5 Find $\bar{z}$ $= \dfrac{M_{xy}}{M} = \dfrac{\frac{8-\pi}{2}}{2\pi - 2} = \boxed{\dfrac{8-\pi}{4\pi-4}}$

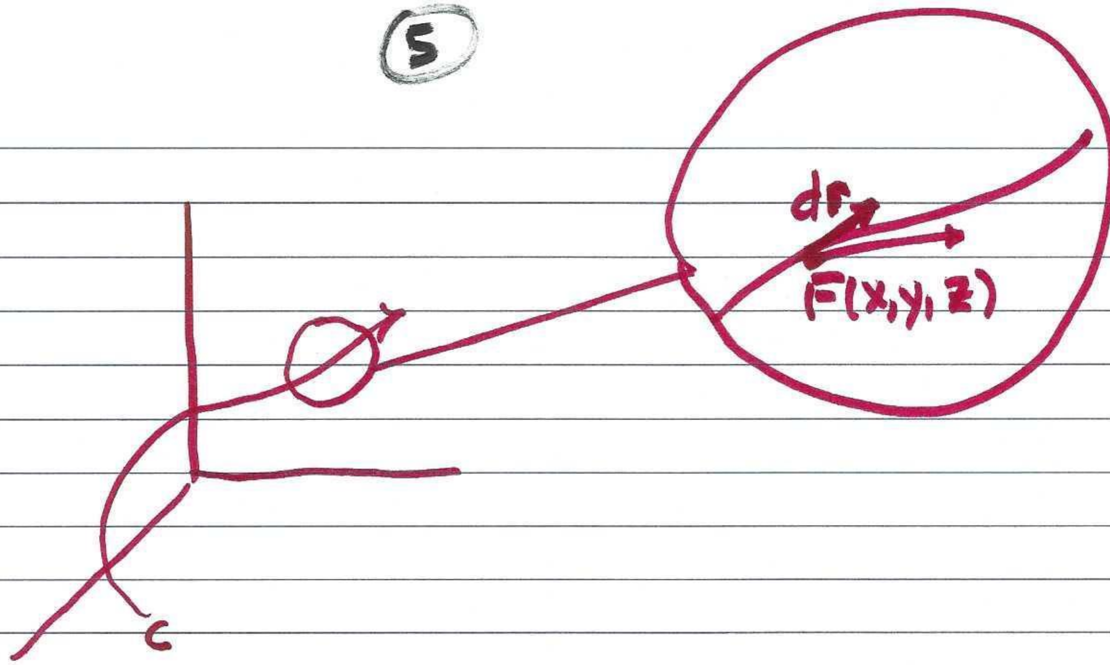
$= 0.567$

Vector Fields

$\mathbb{F} : \mathbb{R}^3 \to \mathbb{R}^3$

Heat, Elec, Mag





$$L = \int_C \mathbf{F} \cdot d\mathbf{r}$$

$$\mathbf{F}(x,y,z) = M(x,y,z)\,\hat{i} + N(x,y,z)\,\hat{j} + P(x,y,z)\,\hat{k}$$

$$d\mathbf{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$$

$$\mathbf{F} \cdot d\mathbf{r} = M\,dx + N\,dy + P\,dz$$

⑥

force $\mathbb{F}(x,y,z) = (y - x^2)\hat{i} + (z - y^2)\hat{j} + (x - z^2)\hat{k}$

displace $\mathbb{r}(t) = t\hat{i} + t^2\hat{j} + t^3\hat{k}$

$\mathbb{F}(t) = (t^2 - t^2)\hat{i} + (t^3 - t^4)\hat{j} + (t - t^6)\hat{k}$

dot

$= (t^3 - t^4)\hat{j} + (t - t^6)\hat{k}$

$\mathbb{r}'(t) = \hat{i} + 2t\hat{j} + 3t^2\hat{k}$

$\rightarrow |\mathbb{r}'(t)| = \sqrt{1 + 4t^2 + 9t^4}$

Path goes from origin to $(1,1,1)$

Note $0 \le t \le 1$

$$\mathbb{F}(t) \cdot \frac{d\mathbb{r}}{dt} = \left[(t^3 - t^4)\hat{j} + (t - t^6)\hat{k}\right] \cdot \left[\hat{i} + 2t\hat{j} + 3t^2\hat{k}\right]$$

$$\mathbb{F}(t) \cdot \frac{d\mathbb{r}}{dt} = 2t(t^3 - t^4) + 3t^2(t - t^6)$$

$$L = \int_0^1 \left(2t^4 - 2t^5 + 3t^3 - 3t^8\right) dt$$

$$= \left[\frac{2t^5}{5} - \frac{t^6}{3} + \frac{3t^4}{4} - \frac{t^9}{3}\right]_0^1$$

$$= \underline{\frac{2}{5} - \frac{1}{3} + \frac{3}{4} - \frac{1}{3}} = \underline{0.48\overline{3}}$$

#19 $\mathbf{F}(x,y,z) = xy\,\hat{i} + y\,\hat{j} - yz\,\hat{k}$

$\mathbf{r}(t) = t\,\hat{i} + t^2\,\hat{j} + t\,\hat{k} \qquad 0 \le t \le 1$

$\mathbf{F}(t) = t^3\,\hat{i} + t^2\,\hat{j} - t^3\,\hat{k}$

$$\mathbf{r}'(t) = \hat{i} + 2t\hat{j} + \hat{k}$$

$$\mathbf{F}(t)\cdot\mathbf{r}'(t) = t^3 + 2t^3 - t^3 = 2t^3$$

$$\int_0^1 \mathbf{F}(t)\cdot\mathbf{r}'(t)\,dt = \int_0^1 2t^3\,dt = \left[\frac{t^4}{2}\right]_0^1 = \frac{1}{2}$$

Fluid has velocity $= x\hat{i} + z\hat{j} + y\hat{k}$

Path is helix : $\mathbf{r}(t) = \cos t\,\hat{i} + \sin t\,\hat{j} + t\hat{k}$

$0 \le t \le \frac{\pi}{2}$

Flow $:= \int \mathbf{F}(t)\cdot\mathbf{r}'(t)\,dt$

$$\mathbf{F}(t) = \cos t\,\hat{i} + t\hat{j} + \sin t\,\hat{k}$$

$$\mathbf{r}'(t) = -\sin t\,\hat{i} + \cos t\,\hat{j} + \hat{k}$$

$$\mathbf{F}(t)\cdot\mathbf{r}'(t) = -\sin t\cos t + t\cos t + t\sin t$$

$$\text{Flow} = \int_0^{\pi/2} (t\sin t + t\cos t - \sin t\cos t)\,dt$$

$$= \left[\frac{\cos^2 t}{2} + t\sin t\right]_0^{\pi/2} = \frac{\pi}{2} - \left(\frac{1}{2}\right) = \frac{\pi - 1}{2}$$

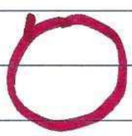
Circulation Integral (is line integral where start & stop points are identical)

Ex: $\mathbb{F} = (x-y)\,\hat{i} + x\,\hat{j}$.

$\mathbb{r}(t) = \cos t\,\hat{i} + \sin t\,\hat{j} \qquad 0 \le t \le 2\pi$

$\mathbb{F}(t) = (\cos t - \sin t)\,\hat{i} + \cos t\,\hat{j}$

$\mathbb{r}'(t) = -\sin t\,\hat{i} + \cos t\,\hat{j}$



$\mathbb{F}(t) \cdot \mathbb{r}'(t) = -\sin t\cos t + \sin^2 t + \cos^2 t$

$= 1 - \sin t\cos t$

$$\text{Circ} = \oint_0^{2\pi} (1 - \sin t\cos t)\,dt$$

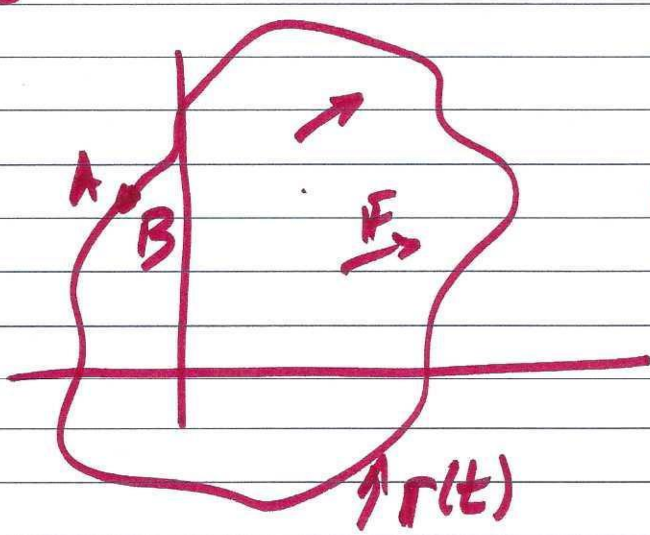
$$\oint_0^{2\pi} (1-\sin t\cos t)\,dt = \left[t - \frac{\sin^2 t}{2}\right]_0^{2\pi} = \boxed{2\pi}$$

Def^n: F is vector field

"curl" of F $:= \nabla \times F$

"divergence" of F $:= \nabla \cdot F$

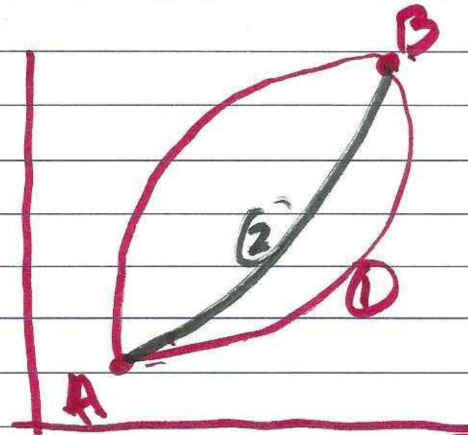
curl is an indicator of "conservative fields"



$\nabla \times F = 0$

implies

$\oint F \cdot dr = 0$



$$\int_{①A}^{B} F\cdot dr + \int_{②B}^{A} F\cdot dr = 0$$

$$\downarrow$$

$$\int_{A\,①}^{B} F\cdot dr = \int_{A\,②}^{B} F\cdot dr$$

Ex: $\mathbb{F}(x,y) = (x-y)\,\hat{i} + x\,\hat{j}$

$$\nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x-y & x & 0 \end{vmatrix}$$

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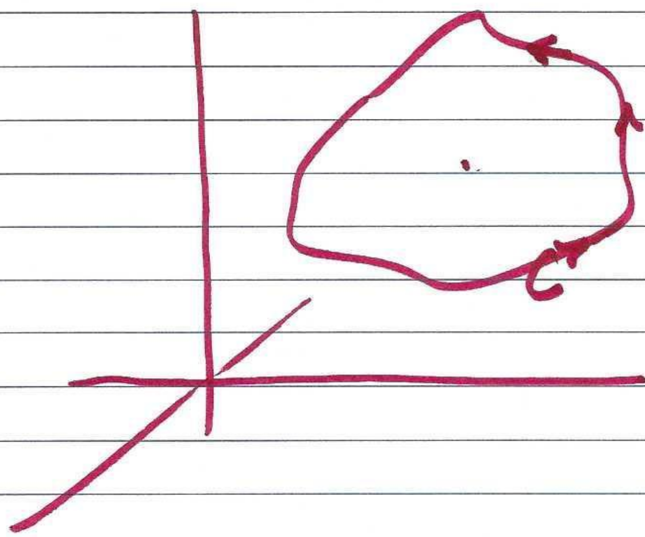
$$\left| \cdot \right| = 0 - 0 + (1+1) = 2$$



$\mathbb{F} \cdot d\mathbf{r}$

$\underline{\mathbb{F}} \cdot \hat{n}\, dt$