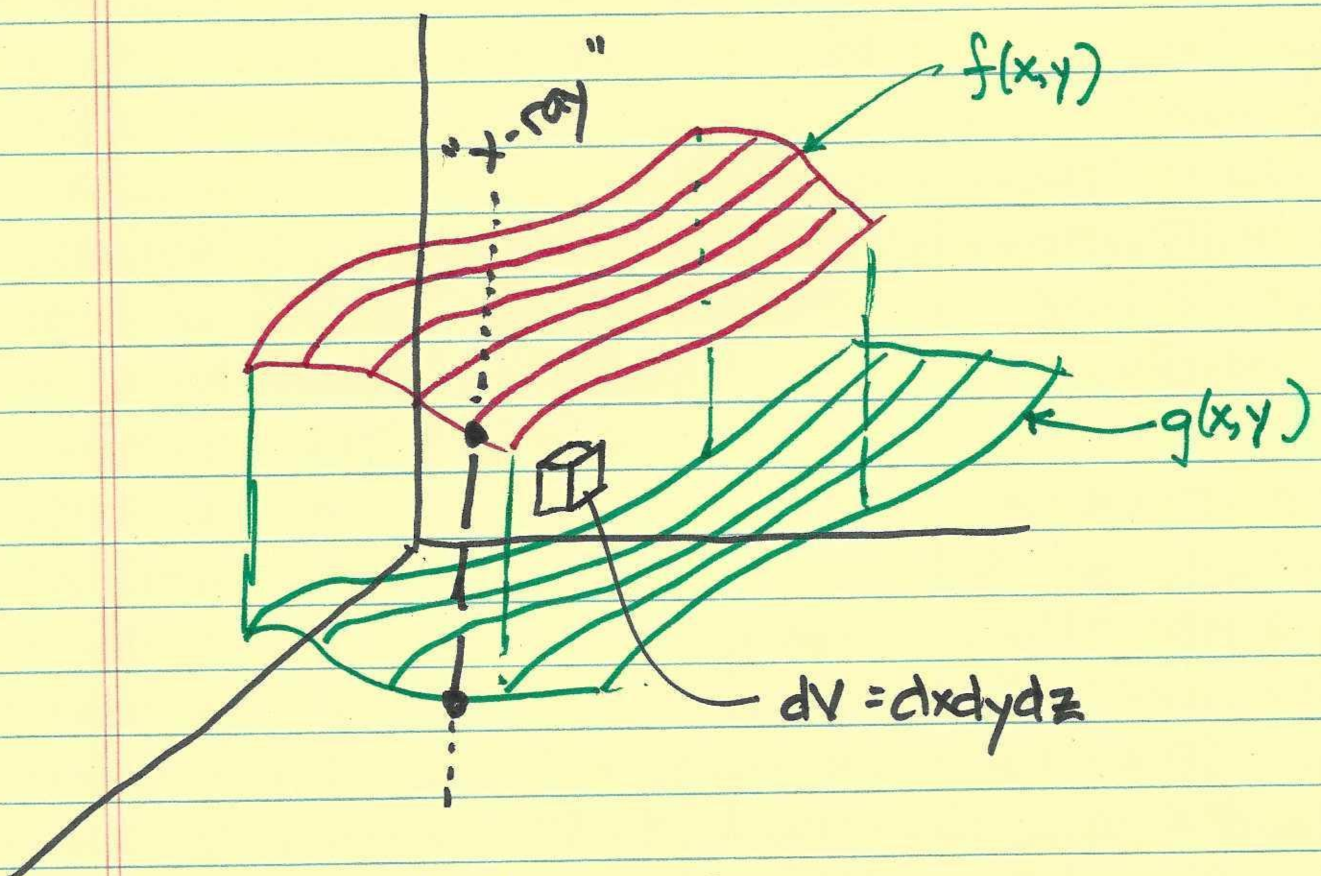
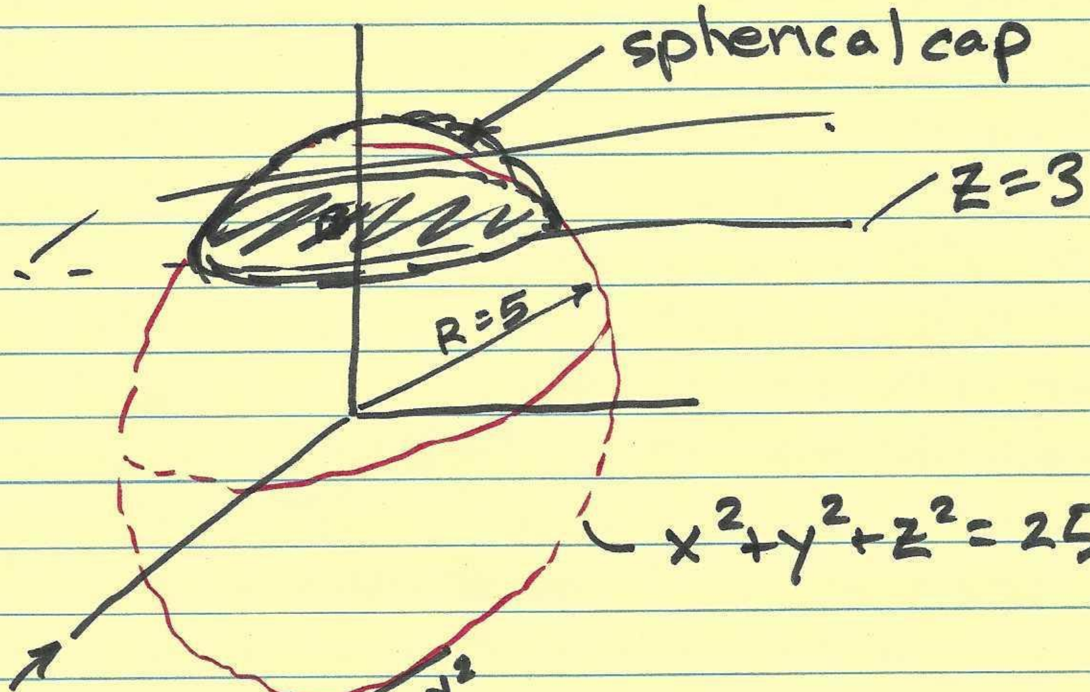
Triple Integrals (§15.5)

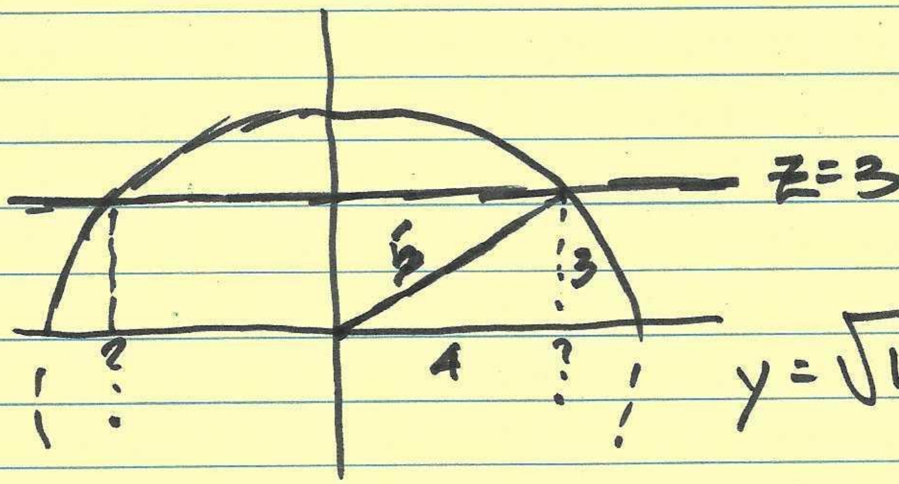


$$V = \int_x \int_y \int_z dz\,dy\,dx$$



$$\int_{-4}^{+4}\int_{-\sqrt{16-x^2}}^{+\sqrt{16-x^2}}\int_{3}^{\sqrt{25-x^2-y^2}} dz\,dy\,dx$$

side view



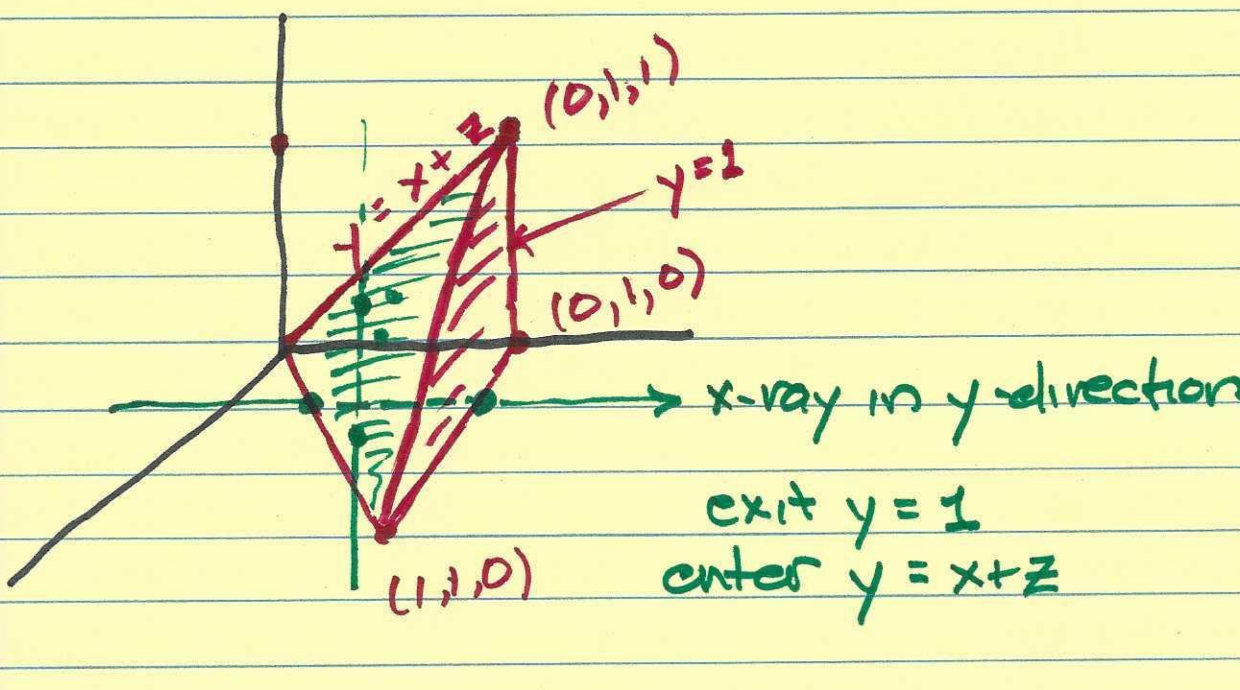
$y=\sqrt{16-x^2}$

$$V = \int_{-4}^{4} \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} \int_{3}^{\sqrt{25-x^2-y^2}} (1)\, dz\, dy\, dx$$

do z:
$$V = \int_{-4}^{4} \int_{-\sqrt{16-x^2}}^{\sqrt{16+x^2}} \left(\sqrt{25-x^2-y^2} - 3\right) dy\, dx$$

integration problem

Tetrahedron



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$$V = \int_0^1 \int_0^{1-x} \int_{x+z}^{1} dy\,dz\,dx$$

$$\frac{1}{2} - x + \frac{3}{2}x^2$$

$$= \int_0^1 \int_0^{1-x} (1-x-z)\,dz\,dx$$

$$= \int_0^1 \left(z - xz - \frac{z^2}{2}\right)\Big|_0^{1-x} dx$$

$$= \int_0^1 \left((1-x) - (x-x^2) - \frac{(1-x)^2}{2}\right) dx$$

$$= \int_0^1 \left(\frac{1}{2} - x + \frac{3}{2}x^2\right) dx$$

correct ans is 1/6

$$= \left[\frac{x}{2} - \frac{x^2}{2} + \cancel{\frac{3}{2}}\,\frac{x^3}{\cancel{3}}\right]_0^1 = \frac{1}{2} - \frac{1}{2} + \frac{1}{2} = \cancel{\frac{1}{2}}$$

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to find boundary of intersection:

Set two paraboloid eqn's equal:

$$8 - x^2 - y^2 = \underbrace{x^2 + 3y^2}$$

$$8 - 2x^2 - 4y^2 = 0$$

controls x & y

$$4 - x^2 - 2y^2 = 0 \text{ or } \underline{x^2 + 2y^2 = 4}$$

eqn of intersection

$$V = \int_{-2}^{2} \int_{-\sqrt{\frac{4-x^2}{2}}}^{+\sqrt{\frac{4-x^2}{2}}} \int_{x^2+3y^2}^{8-x^2-y^2} dz\,dy\,dx$$

Task is find y-limits in terms of x:

$$x^2 + 2y^2 = 4$$

$$y^2 = \frac{4-x^2}{2} \Rightarrow y = \pm\sqrt{\frac{4-x^2}{2}}$$

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$$V = \int_{-2}^{2} \int_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}} \int_{x^2+3y^2}^{8-x^2-y^2} dz\,dy\,dx$$

$$V = \int_{-2}^{2} \int_{\sim}^{\sim} \left[(8-x^2-y^2)-(x^2+3y^2)\right] dy\,dx$$

$$= \int_{-2}^{2} \int_{\sim}^{\sim} (8-2x^2-4y^2)\,dy\,dx$$

$$= \int_{-2}^{2} \left[8y-2x^2y-4\frac{y^3}{3}\right]_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}} dx$$

$$= \int_{-2}^{2} \left[8\cdot 2\sqrt{\frac{4-x^2}{2}} - 4x^2\sqrt{\frac{4-y^2}{2}} - \frac{2\cdot 4}{3}\left(\frac{4-x^2}{2}\right)^{3/2}\right] dx$$

$$= \int_{-2}^{2} \left(\sqrt{\frac{4-x^2}{2}} \left(16 - x^2 - \frac{8}{3}\left(\frac{4-x^2}{2}\right) \right) \right) dx$$

$$= \int_{-2}^{2} \left(\frac{4-x^2}{2}\right)^{1/2} \cdot \left(\frac{32}{3} - \frac{7}{3}x^2 \right) dx$$

$\rightarrow$ see book for substitution $x = 2\sin u$

$\Rightarrow$ $V = 8\pi\sqrt{2}$

3-D average value of $f(x,y,z)$ is:

$$\bar{f} = \frac{\iiint f(x,y,z)\,dx\,dy\,dz}{\iiint dx\,dy\,dz}$$

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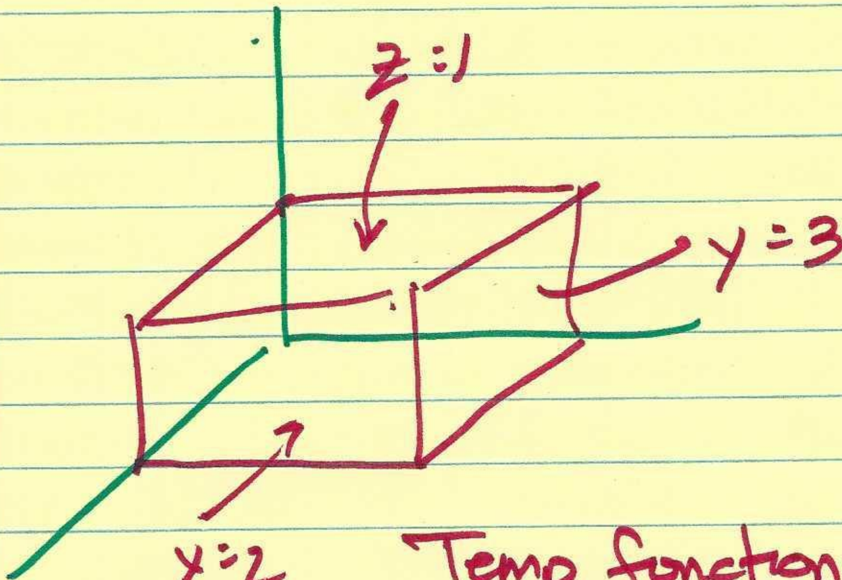
Find average value of temperature over rectangular solid:



Temp function is e^{x+y}

$$\bar{T} = \frac{\int_0^3\int_0^2\int_0^1 e^{x+y}\,dz\,dx\,dy}{\int_0^3\int_0^2\int_0^1 dz\,dx\,dy}$$

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$$\int_0^3\int_0^2\int_0^1 e^{x+y}\,dz\,dx\,dy = \int_0^3\int_0^2 [z]_0^1 e^{x+y}\,dx\,dy$$

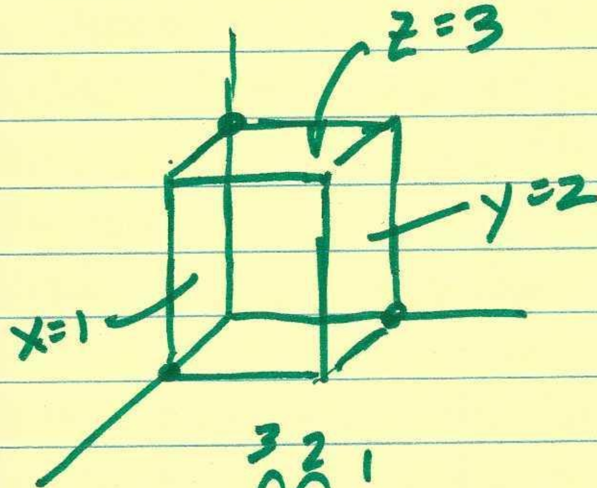
$$= \int_0^3\int_0^2 e^{x+y}\,dx\,dy = \int_0^3 \left[e^{x+y}\right]_0^2 dy$$

$$= \int_0^3 \left(e^{2+y} - e^{y}\right) dy = \left[e^{2+y} - e^{y}\right]_0^3 =$$

$$(e^5 - e^3) - (e^2 - 1) = \underline{e^5 - e^3 - e^2 + 1}$$

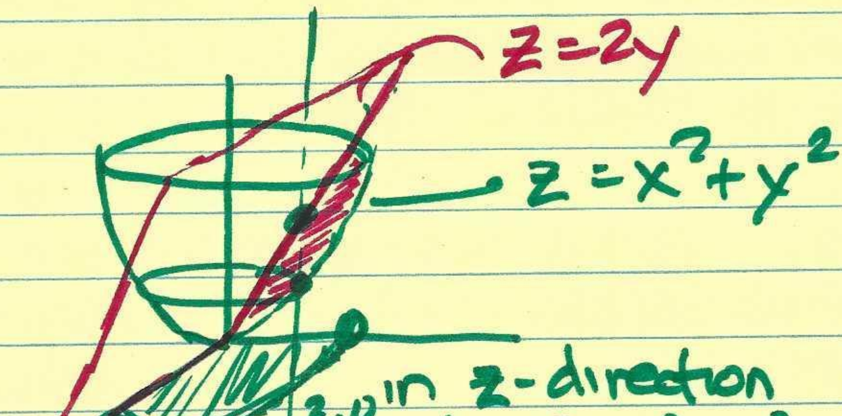
so $\overline{T} = \frac{1}{6}\left(e^5 - e^3 - e^2 + 1\right)$

p. 929
#2



$$V = \int_0^3\int_0^2\int_0^1 dx\,dy\,dz$$

$$\int_0^3\int_0^1\int_0^2 \,,\quad \int_0^2\int_0^1\int_0^3 \,,\quad \ldots$$



$x^2-2y+y^2=0$

in z-direction
enter @ x^2+y^2
exit @ $2y$

height (z) of figure is when $x^2+y^2=2y$

$$x^2-2y+y^2=0 \Rightarrow x=\pm\sqrt{y^2-2y}$$

$$V=\int_0^2\int_{-\sqrt{y^2-2y}}^{+\sqrt{y^2-2y}}\int_{x^2+y^2}^{2y} dz\,dx\,dy$$

y-limits are 0 & solution to $y^2=2y$

$$(y-2)(y)=0$$

#9 $$\int_1^e\int_1^{e^2}\int_1^{e^3}\frac{1}{xyz}\,dx\,dy\,dz$$

$$\int_1^e\int_1^{e^2}\frac{1}{yz}\Big[\ln x\Big]_1^{e^3}dy\,dz=\int_1^e\int_1^{e^2}\frac{1}{yz}(3-0)\,dy\,dz$$

$$=3\int_1^e\int_1^{e^2}\frac{1}{yz}\,dy\,dz=3\int_1^e\frac{1}{z}\Big[\ln y\Big]_1^{e^2}=6\int_1^e\frac{dz}{z}$$

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#18 $$\int_0^1 \int_1^{\sqrt{e}} \int_1^e s e^{s} \ln r \frac{(\ln t)^2}{t}\, dt\, dr\, ds$$

$$= \int_0^1 \int_1^{\sqrt{e}} \int_1^e s e^{s} \ln r \frac{(\ln t)^2}{t}\, dt\, dr\, ds$$

Look @ $\int_1^e \frac{(\ln t)^2}{t}\, dt \qquad d(\ln t) = \frac{dt}{t}$

$$= \left[\frac{(\ln t)^3}{3}\right]_1^e = \frac{1}{3}$$

$$= \frac{1}{3} \int_0^1 \int_1^{\sqrt{e}} s e^{s} \ln r\, dr\, ds = \frac{1}{3} \int_0^1 s e^{s} \left[r \ln r - r\right]_1^{\sqrt{e}} ds$$

$$\left(\frac{1}{2}\sqrt{e}-\sqrt{e}\right)-\left(-1\right) = \sqrt{e}\left(-\frac{1}{2}\right)+1$$

$$1-\frac{\sqrt{e}}{2}$$

$$= \frac{1}{3}\left(1-\frac{\sqrt{e}}{2}\right)\int_0^1 se^s\,ds$$

separately $\int se^s\,ds = se^s - \int e^s\,ds = se^s - e^s$

eval: $\left[se^s - e^s\right]_0^1 =$

$$(e-e)-(-1) = 1$$

Finally $\iiint dt\,dr\,ds = \frac{1}{3}\left(1-\frac{\sqrt{e}}{2}\right)$

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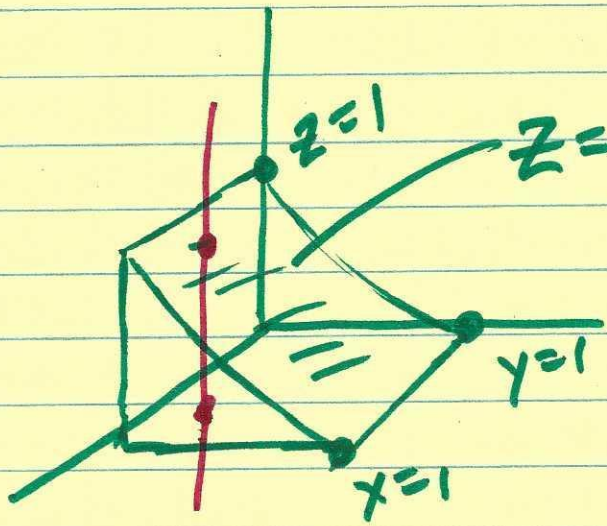

$$V = \int_0^1 \int_0^1 \int_0^{y^2} dz\,dy\,dx$$

$$= \int_0^1 \int_0^1 y^2\,dy\,dx = \int_0^1 \left[\frac{y^3}{3}\right]_0^1 dx =$$

$$\frac{1}{3}\int_0^1 dx = \frac{1}{3}$$

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$$V = \int_0^1 \int_{-1}^1 \int_0^{y^2} dz\,dy\,dx$$

$$= \int_0^1 \int_{-1}^1 y^2\,dy\,dx = \int_0^1 \left[\frac{y^3}{3}\right]_{-1}^1 dx =$$

$$\frac{2}{3}\int_0^1 dx = \frac{2}{3}$$