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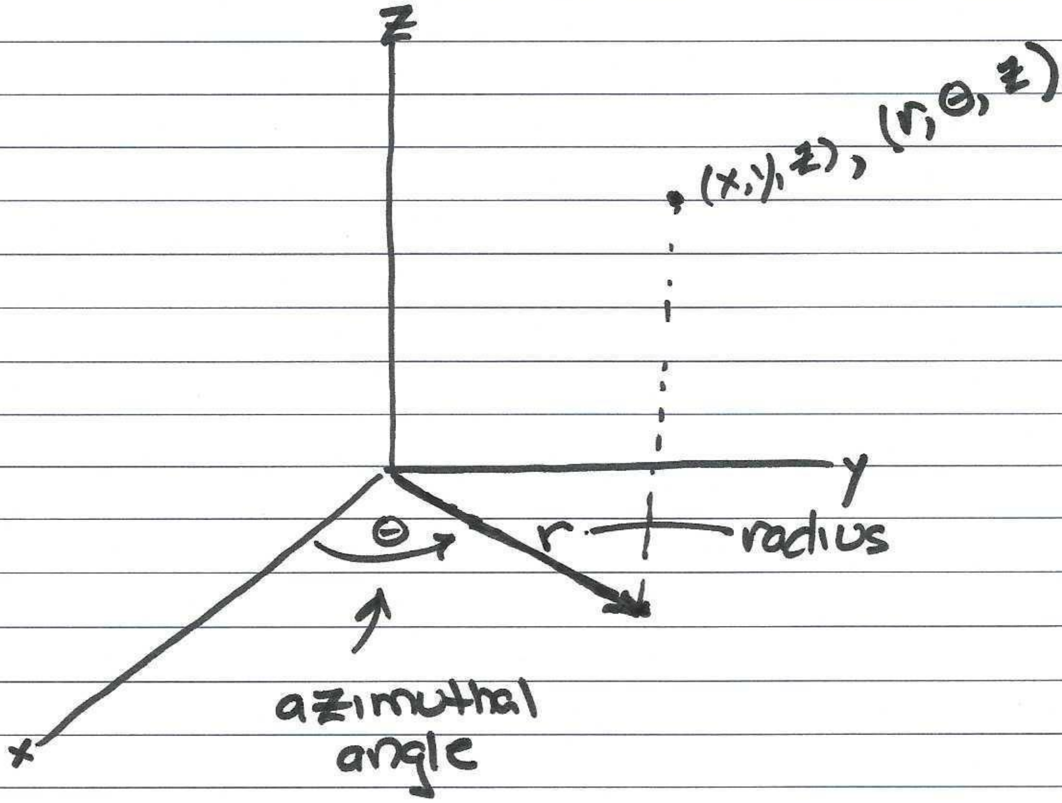
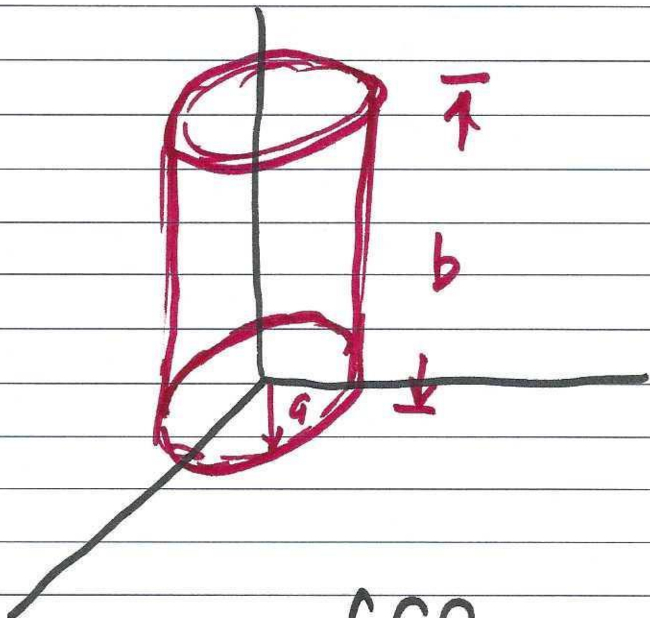
Cylindrical Coördinates



Volume element : $dV = dx\,dy\,dz$ (cartesian)

$dV = r\,dr\,d\theta\,dz$ (cylindrical)

What is volume of a cylinder of radius a and height b aligned with z-axis



$$V = \iiint_{\theta\, z\, r} r\,dr\,dz\,d\theta$$

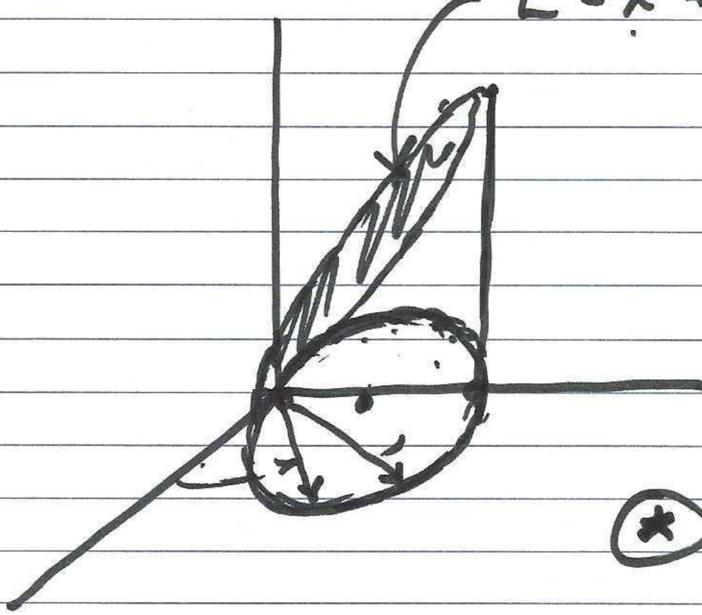
$$= \int_0^{2\pi}\int_0^{b}\int_0^{a} r\,dr\,dz\,d\theta =$$

$$= \int_0^{2\pi}\int_0^{b}\left[\frac{r^2}{2}\right]_0^{a} dz\,d\theta$$

③

$$\text{Vol} = \int_0^{2\pi}\int_0^{b} \frac{a^2}{2}\,dz\,d\theta = \int_0^{2\pi} \frac{a^2 b}{2}\,d\theta =$$

$$= 2\pi \cdot \frac{a^2 b}{2} = \underline{\pi a^2 b}$$



$z = x^2 + y^2 \; (= r^2)$

$x^2 + (y-1)^2 = 1$

$\underbrace{x^2+y^2}-2y+1 = 1$

$\circledast \; r^2 - 2r\sin\theta = 0$

$r(\underline{r - 2\sin\theta}) = 0$

$r = 2\sin\theta$ ✓

$x = r\cos\theta$
$y = r\sin\theta$
$z = z$

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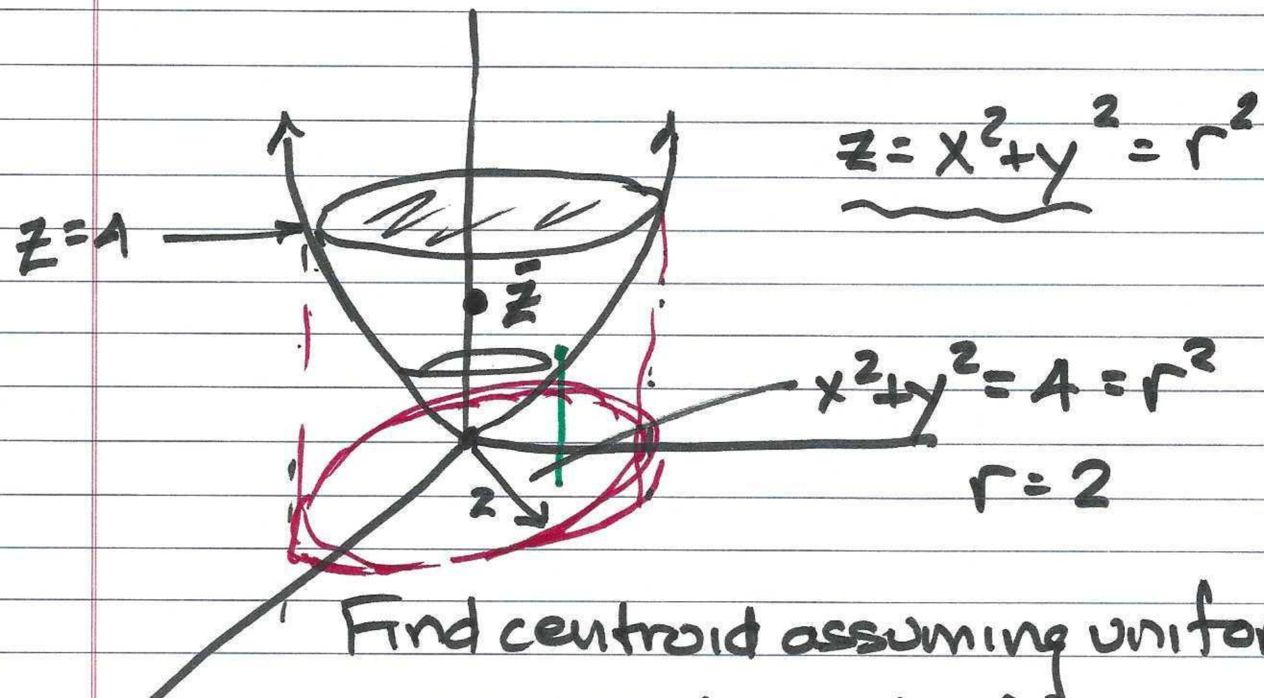
$$\text{Vol} = \int_{\theta=0}^{\pi}\int_{r=0}^{2\sin\theta}\int_{z=0}^{r^2} dz\, r\,dr\,d\theta$$

$$= \int_0^{\pi}\int_0^{2\sin\theta} \left[z\right]_0^{r^2} r\,dr\,d\theta = \int_0^{\pi}\int_0^{2\sin\theta} r^3\,dr\,d\theta$$

$$= \int_0^{\pi}\left[\frac{r^4}{4}\right]_0^{2\sin\theta} d\theta = \frac{1}{4}\int_0^{\pi}(2\sin\theta)^4\,d\theta$$

$$= 8\int_0^{\pi}\sin^4\theta\,d\theta$$ OK... look up or use Wolfram/Maple



Find centroid assuming uniform mass density ρ $(=1)$

$$V = \int_0^{2\pi}\int_0^{2}\int_{r^2}^{4} dz\, r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{2} \Big[z\Big]_{r^2}^{4} r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{2} (4-r^2)\, r\,dr\,d\theta$$

⑥

$$= \int_0^{2\pi}\int_0^2 (4r - r^3)\,dr\,d\theta =$$

$$= \int_0^{2\pi} \left(2r^2 - \frac{r^4}{4}\right)\Big|_0^2 d\theta = \int_0^{2\pi} (8-4)\,d\theta$$

$$= 4\int_0^{2\pi} d\theta = \boxed{8\pi} = \text{Vol.}$$

Moment relative to xy plane M_{xy}

weighting factor

$$M_{xy} = \int_0^{2\pi}\int_0^2\int_{r^2}^4 z\,dz\,r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^2 \left[\frac{z^2}{2}\right]_{r^2}^4 r\,dr\,d\theta$$

(b)

$$M_{xy} = \int_0^{2\pi}\int_0^{2}\left(8-\frac{r^4}{2}\right) r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{2}\left(8r-\frac{r^5}{2}\right) dr\,d\theta$$

$$= \int_0^{2\pi}\left(4r^2-\frac{r^6}{12}\right)\Big|_0^2 \,d\theta$$

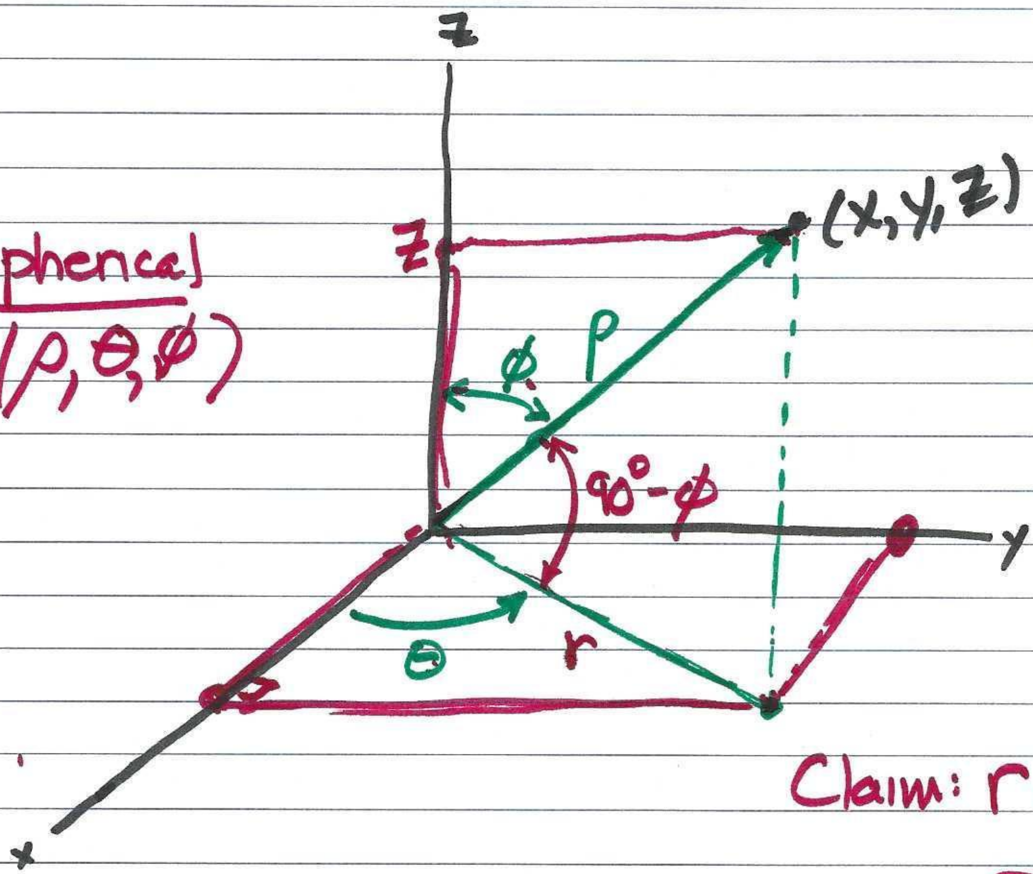
$$= \int_0^{2\pi}\left(16-\frac{64}{12}\right) d\theta = 10.33\int_0^{2\pi} d\theta$$

$$M_{xy} = (10.33)(6.28)$$

So... $\bar{z} = \dfrac{M_{xy}}{M} = \dfrac{64.8724}{25.12} = 2.58$

Spherical

(ρ, θ, ϕ)



Claim: $r = \rho \cos(90 - \phi)$

$r = \rho \sin\phi$

$$\begin{cases} x = \rho \sin\phi \cos\theta \\ y = \rho \sin\phi \sin\theta \\ z = \rho \cos\phi \end{cases}$$

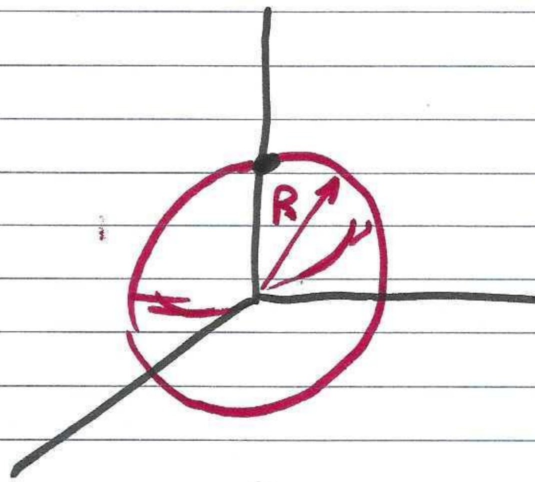
$$\rho = \sqrt{x^2 + y^2 + z^2}$$

Volumetric Mag Factor (jacobian)

$\underline{\rho^2 \sin\phi}$

$$\iiint_R \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

Vol of sphere



ρ from 0 to R

θ from 0 to 2π

ϕ from 0 to π

$$\text{Vol of sphere} = \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi} \int_{\rho=0}^{R} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

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$$= \int_0^{2\pi} \int_0^{\pi} \left[\frac{\rho^3}{3}\right]_0^R \sin\phi \, d\phi \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi} \left(\frac{R^3}{3}\right) \sin\phi \, d\phi \, d\theta$$

$$= \frac{R^3}{3} \int_0^{2\pi} \int_0^{\pi} \sin\phi \, d\phi \, d\theta$$

$$= \frac{R^3}{3} \int_0^{2\pi} \left[-\cos\phi\right]_0^{\pi} d\theta$$

$$= \frac{R^3}{3} \int_0^{2\pi} (1+1) \, d\theta = \frac{2R^3}{3} \int_0^{2\pi} d\theta =$$

$$\frac{2R^3}{3} \left[\theta\right]_0^{2\pi} = \boxed{\frac{4\pi R^3}{3}}$$

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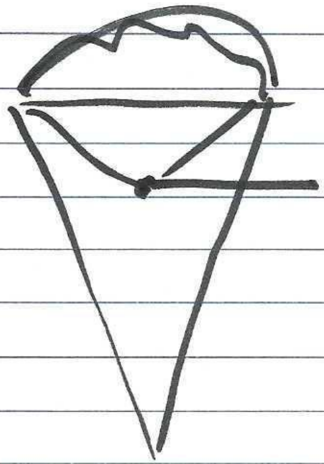
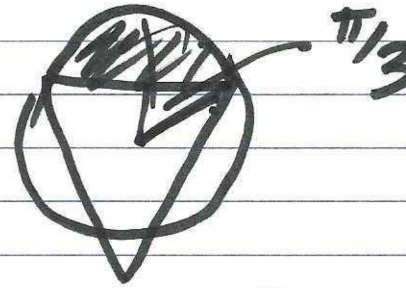
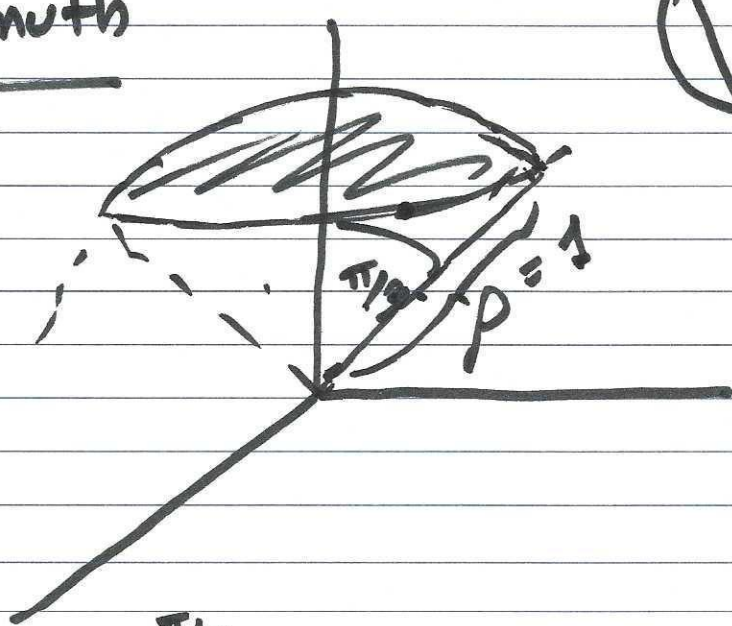
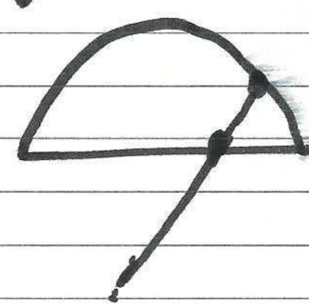


ice cream cone

ϕ = declination
θ = azimuth







$$Vol = \int_0^{2\pi}\int_0^{\pi/3}\int_0^{1} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

(10)

$$= \int_0^{2\pi}\int_0^{\pi/3}\left[\frac{\rho^3}{3}\right]_0^1 \sin\phi \, d\phi \, d\theta$$

$$= \frac{1}{3}\int_0^{2\pi}\int_0^{\pi/3} \sin\phi \, d\phi \, d\theta = \frac{1}{3}\int_0^{2\pi}\left[-\cos\phi\right]_0^{\pi/3} d\theta$$

$$= \frac{1}{3}\int_0^{2\pi}\left(-\frac{1}{2}+1\right)d\theta = \frac{1}{6}\int_0^{2\pi} d\theta = \frac{1}{6}\left[\theta\right]_0^{2\pi} = \boxed{\frac{\pi}{3}}$$

Jacobian is the volume magnification factor

$$\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} = \underline{\rho^2 \sin\phi}$$

$$\iiint f(x,y,z)\,dx\,dy\,dz = \iiint f(x(\rho,\phi,\theta), y(\ \ ), z(\ \ ))\,\rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$

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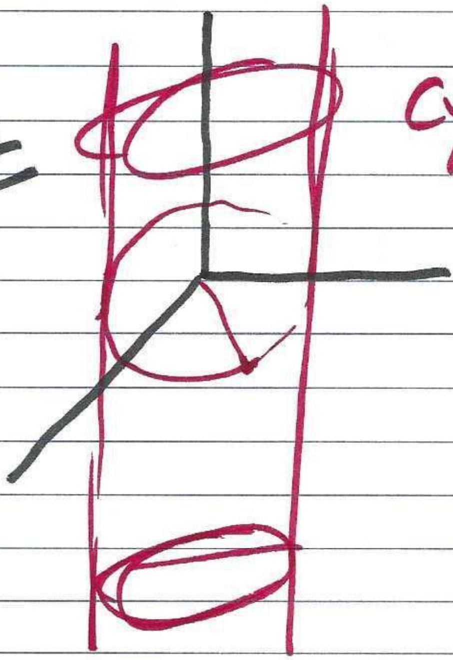
1

$r=2$



Cylinder (infinite)
radius 2

$r=\sqrt{x^2+y^2}$

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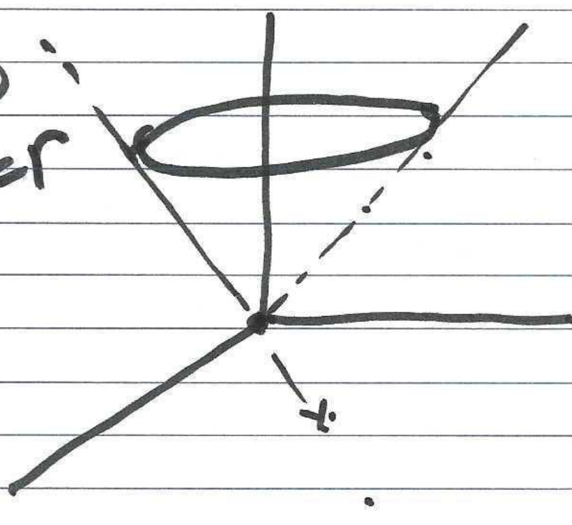
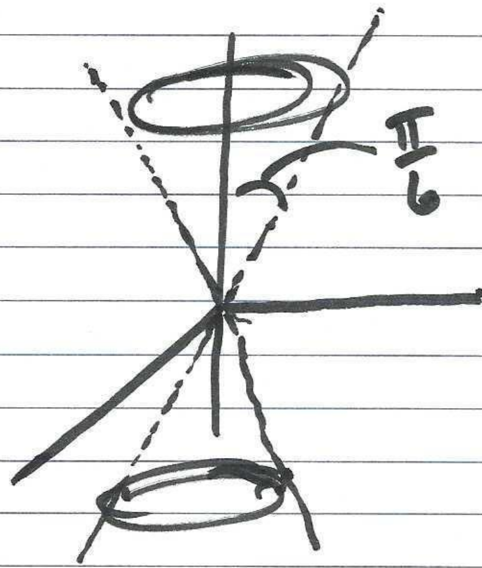
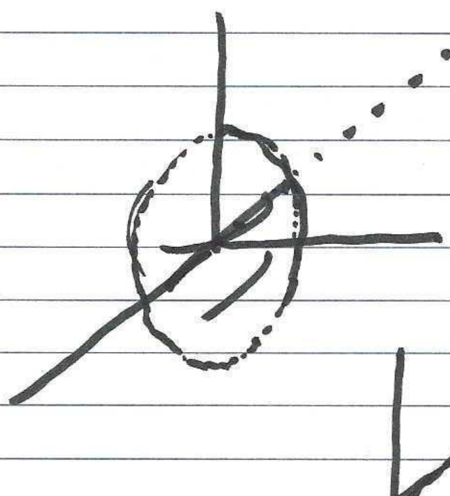
$z=r$



#14 $\phi = \frac{\pi}{6}$



#20



$$\begin{cases} 0 \le \rho \le 1 \\ \frac{\pi}{2} \le \phi \le \pi \\ 0 \le \theta \le \pi \end{cases}$$