A beach cabana is to be designed for minimum surface area (proxy for material cost) that encloses 250 cubic feet. The configuration is to be the exterior surface of a rectangular solid. There is no floor and no front. The ends are to be square (see drawing). Find the dimensions that minimize surface area.

Let L be the length, and W be the common value of the width and height (square ends).

Volume $= 250 = LW^2$. *Surface area* $A(L,W)$ *is* $2LW$ *(back & top)* $+2W^2$ *(ends). Set* $L = \frac{250}{W^2}$, *then* $A(L,W) = A(W) = \frac{500}{W} + 2W^2$. *Then* $A'(W) = \frac{-500}{W^2} + 4W = 0$ *gives a critical point. This simplifies to* $4W^3 = 500$, *or* $W^3 = 125$, *so* $W = 5$. *Checking* $A''(5) = \frac{1000}{5^3} + 4 > 0$ *we conclude* $W = 5$ *and then* $L = 10$ *satisfies the volume condition and gives minimum surface area.*

What if the sides did not have to be square? Then $WLH = 250$. *Eliminate* L: $L = \frac{250}{WH}$. *Then the objective function is (top)* $\frac{250}{WH} \cdot W$ *+(back)* $\frac{250}{WH} \cdot H +$ *(sides)* $2WH$. *Simplified this becomes* $\frac{250}{H} + \frac{250}{W} + 2HW$. *So this time* $A(H,W) = \frac{250}{H} + \frac{250}{W} + 2HW$. *Then* $A_H = 2W - \frac{250}{H^2} = 0$ *gives* $H^2W = 125$. *By symmetry from* $A_W = 0$, *we find* $HW^2 = 125$, *and dividing we see that* $\frac{H}{W} = 1$, *so* $H = W = 5$. *Calculating* $\Delta(H,W) = A_{HH}A_{WW} - A_{HW}^2$ *at* $(5,5)$ *we get* $\left(\frac{500}{125}\right)\left(\frac{500}{125}\right) - 4 > 0$, *so* $H = W = 5$ *and* $L = 10$ *give the same solution.*

4) The function $z = 1 - \frac{x^2}{4} - \frac{y^2}{9}$ describes an elliptical paraboloid that has its axis along the z-axis, its peak at $z = 1$, and extends indefinitely downward below the xy-plane. Cross sections parallel to the xy plane are ellipses. In the region where $x, y, z \geq 0$ we would like to construct a rectangular box with one corner at the origin and the diametrically opposite corner on the paraboloid. Find the volume of the largest box that can be constructed subject to these conditions.

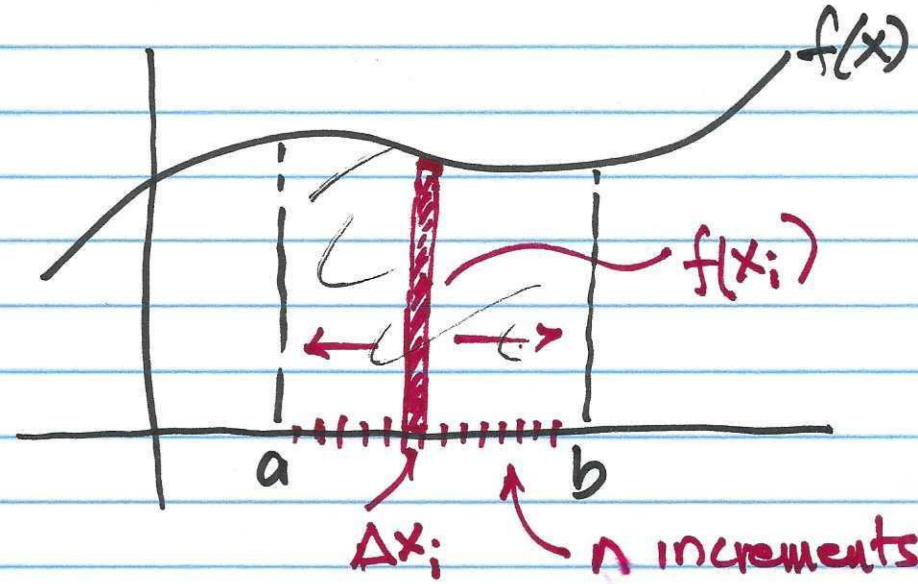
The volume of the box is xyz. *Then* $V(x,y) = xy\left(1 - \frac{x^2}{4} - \frac{y^2}{9}\right) = xy - \frac{x^3y}{4} - \frac{xy^3}{9}$. *This is the objective function to be maximized.* $V_x = y - \frac{3x^2y}{4} - \frac{y^3}{9} = y(1 - \frac{3x^2}{4} - \frac{y^2}{9}) = 0$. *We note* $y = 0$ *cannot be a solution or the box would have no volume. So* $1 - \frac{3x^2}{4} - \frac{y^2}{9} = 0$. *Similarly,* $V_y = x - \frac{x^3}{4} - \frac{xy^2}{3} = x(1 - \frac{x^2}{4} - \frac{y^2}{3}) = 0$, *hence* $(1 - \frac{x^2}{4} - \frac{y^2}{3}) = 0$. *To solve these simultaneously, we rewrite them as* $\frac{3x^2}{4} = 1 - \frac{y^2}{9}$ *and* $\frac{x^2}{4} = 1 - \frac{y^2}{3}$. *Then*

$3\left(\frac{x^2}{4}\right) = 3\left(1 - \frac{y^2}{3}\right)$, *or* $\frac{3x^2}{4} = 3 - y^2$. *It follows that* $1 - \frac{y^2}{9} = 3 - y^2$ *which leads to the quadratic* $18 = 8y^2$ *and finally* $y = \frac{3}{2}$. *Plugging back into the expression for* x, *we find* $x = 1$. *Note* $V_{xx} = -\frac{3xy}{2}$ *and* $V_{yy} = -\frac{2xy}{3}$ *and* $V_{xy} = 1 - \frac{3x^2}{4} - \frac{y^2}{3}$, *so the discriminant* $\Delta(1, \frac{3}{2}) = V_{xx}(1, \frac{3}{2})V_{yy}(1, \frac{3}{2}) - V_{xy}^2(1, \frac{3}{2}) = \left(-\frac{9}{4}\right)(-1) - (1 - \frac{3}{4} - \frac{3}{4})^2 = \frac{9}{4} - \frac{1}{4} = 2 > 0$. *We have a winner.* $V(x,y)$ *has a maximum at* $(1, \frac{3}{2})$. *Specifically,* $z = 1 - \frac{1}{4} - \frac{1}{4} = \frac{1}{2}$. *So the box with sides* 1 *by* $\frac{3}{2}$ *by* $\frac{1}{2}$ *has the maximum volume equal to* $\frac{3}{4}$.

10/29

Double Integrals

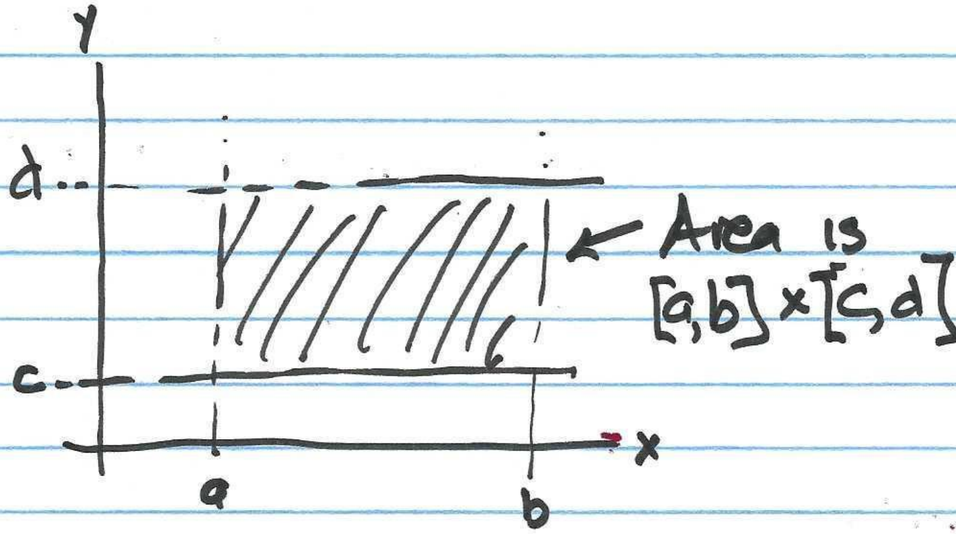
Recall 1-D Riemann integral:

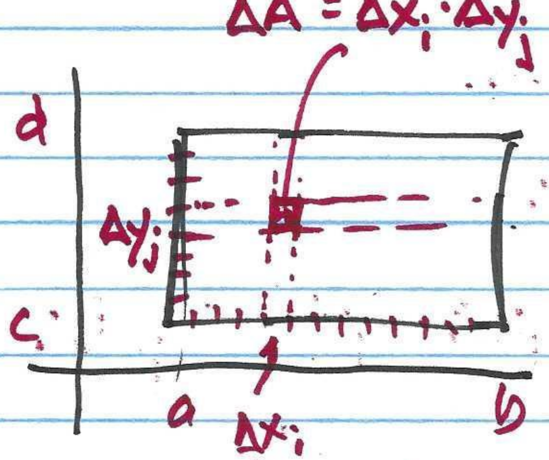


$$\Delta A_i = f(x_i) \cdot \Delta x_i$$

$$\text{Area} \approx \sum_{i=1}^{n} f(x_i) \cdot \Delta x_i$$

$$\lim_{\Delta x_i \to 0} \sum_{i=1}^{n} f(x_i) \cdot \Delta x_i = \int_a^b f(x)\,dx$$





$f(x,y)$ is floating above

$f(x_i, y_j)$ is value selected in grid element

$\Delta x_i, \Delta y_j$

$$\Delta V_{ij} = f(x_i, y_j) \cdot \Delta x_i \cdot \Delta y_j$$

$$V \cong \sum_i \sum_j f(x_i, y_j)\, \Delta x_i \Delta y_j$$

$$\lim_{\Delta x_i, \Delta y_j \to 0} \left(\sum_i \sum_j f(x_i, y_j) \Delta x_i \Delta y_j \right) = \iint_R f(x,y)\,dx\,dy$$

$$R = [a,b] \times [c,d]$$

Fubini's Th^{m}:

$$\iint_R f(x,y)\,dx\,dy = \int_a^b \left[\int_c^d f(x,y)\,dy \right] dx =$$

$$\int_c^d \left[\int_a^b f(x,y)\,dx \right] dy$$

iterated integrals to eval a <u>double</u> integral

$$\int_1^2 \int_0^4 \underline{2xy\,dy}\,dx = \int_1^2 \left[\int_0^4 2xy\,dy \right] dx$$

$$= \int_1^2 \left[\left[xy^2 \right]_0^4 \right] dx = \int_1^2 16x\,dx = \left[8x^2 \right]_1^2 =$$

$$32 - 8 = 24$$

4

$$\int_0^3\int_0^2 (4-y^2)\,dy\,dx = \int_0^3 \left[4y - \frac{y^3}{3}\right]_0^2 dx$$

$$= \frac{16}{3}\int_0^3 dx = \frac{16}{3}\left[x\right]_0^3 = 16$$

$$\int_0^1\int_1^2 xye^x\,dy\,dx = ?$$

$$\int_0^1 xe^x\left[\frac{y^2}{2}\right]_1^2 dx = \int_0^1 \left(\frac{3}{2}\right)xe^x\,dx =$$

$$\frac{3}{2}\int_0^1 xe^x\,dx$$

$$\hookrightarrow\ xe^x - \int e^x\,dx = e^x(x-1)$$

So $\frac{3}{2}\int_0^1 xe^x\,dx = \frac{3}{2}\left(0-(-1)\right) = \frac{3}{2}$

5

$$\iint_R (6y^2 - 2x)\,dA \quad \Big| \quad R: \overset{x}{[0,1]} \times [0,2]$$

$$\int_{y=0}^{2} \int_0^1 (6y^2 - 2x)\,dx\,dy =$$

$$\int_0^2 \left[6xy^2 - x^2\right]_0^1 dy = \int_0^2 (6y^2 - 1)\,dy =$$

$$\left[2y^3 - y\right]_0^2 = 14$$

⑥

$$V = \iint (x^2+y^2)\,dA \qquad [-1,1]\times[-1,1]$$

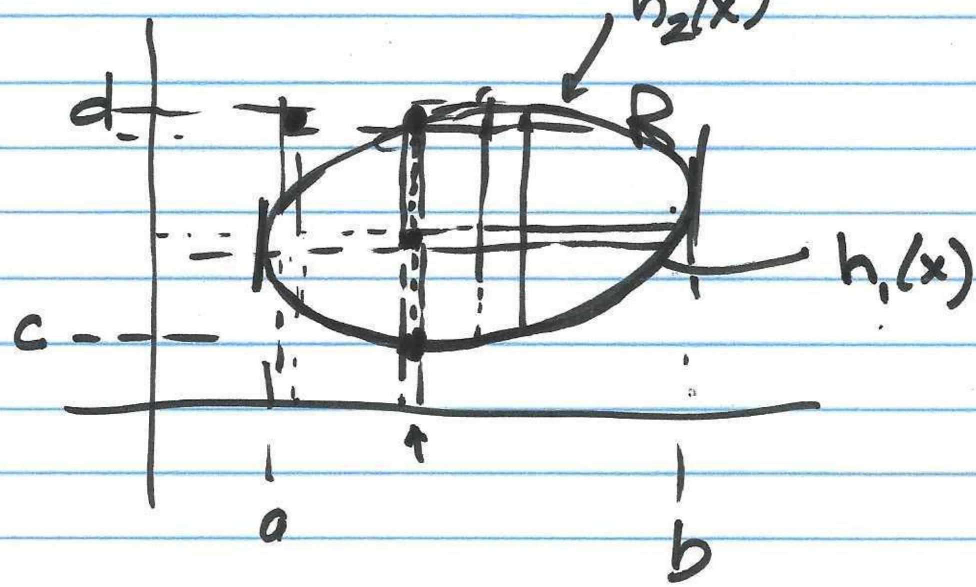
$$\int_{-1}^{1}\int_{-1}^{1}(x^2+y^2)\,dx\,dy = \int_{-1}^{1}\left[\frac{x^3}{3}+xy^2\right]_{-1}^{1}dy$$

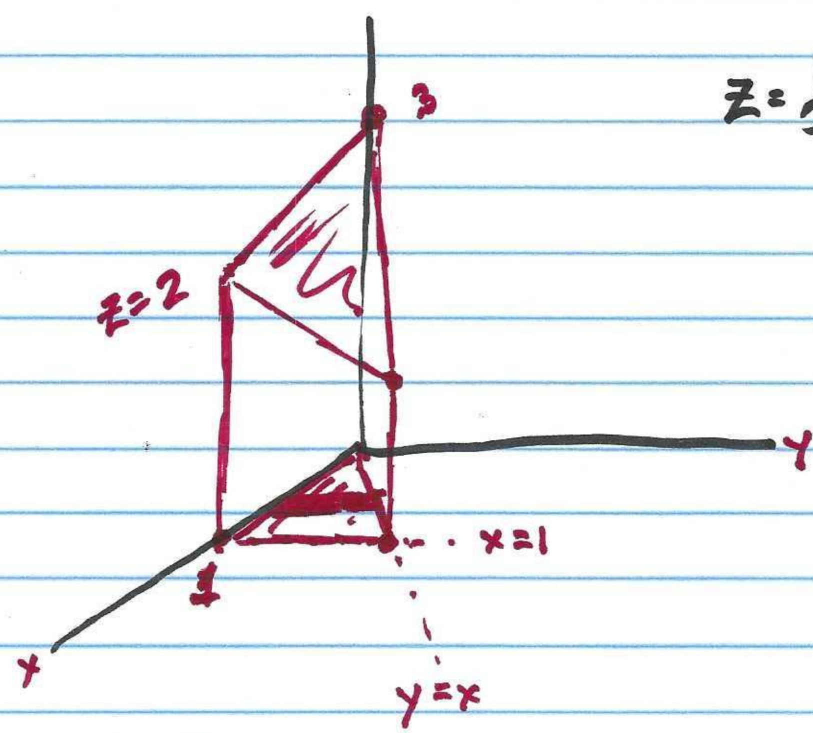
$$= \int_{-1}^{1}\left(\frac{2}{3}+2y^2\right)dy \qquad \underline{\left(\frac{1}{3}+y^2\right)-\left(-\frac{1}{3}-y^2\right)}$$

$$= \left[\frac{2}{3}y+\frac{2y^3}{3}\right]_{-1}^{1} =$$

$$\left[\frac{2}{3}+\frac{2}{3}\right]-\left[-\frac{2}{3}-\frac{2}{3}\right] = 8/3$$

7





$z = f(x,y) = 3-x-y$

$$V = \int_{x=0}^{1}\int_{0}^{x}(3-x-y)\,dy\,dx$$

8

$$V = \int_0^1 \left[3y - xy - \frac{y^2}{2}\right]_0^x dx$$

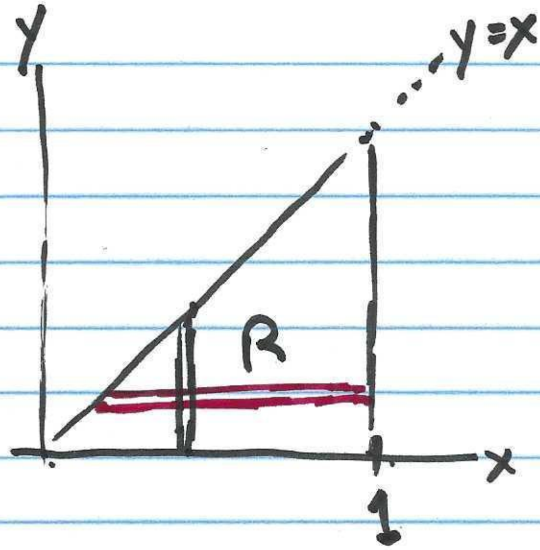
$$= \int_0^1 \left(3x - x^2 - \frac{x^2}{2}\right) dx$$

$$= \int_0^1 \left(3x - \frac{3}{2}x^2\right) dx = \left[\frac{3x^2}{2} - \frac{x^3}{2}\right]_0^1$$

$$\frac{3}{2} - \frac{1}{2} - 0 = 1$$

(9)

Ex.



$$\iint_R \frac{\sin x}{x}\, dA$$

(I) $$\int_0^1 \left(\int_0^x \frac{\sin x}{x}\, dy \right) dx$$

(II) $$\int_0^1 \left(\int_y^1 \frac{\sin x}{x}\, dx \right) dy$$

not doable in elem functions

(10)

Choice I. $\displaystyle\int_0^1\left(\int_0^x \frac{\sin x}{x}\,dy\right)dx =$

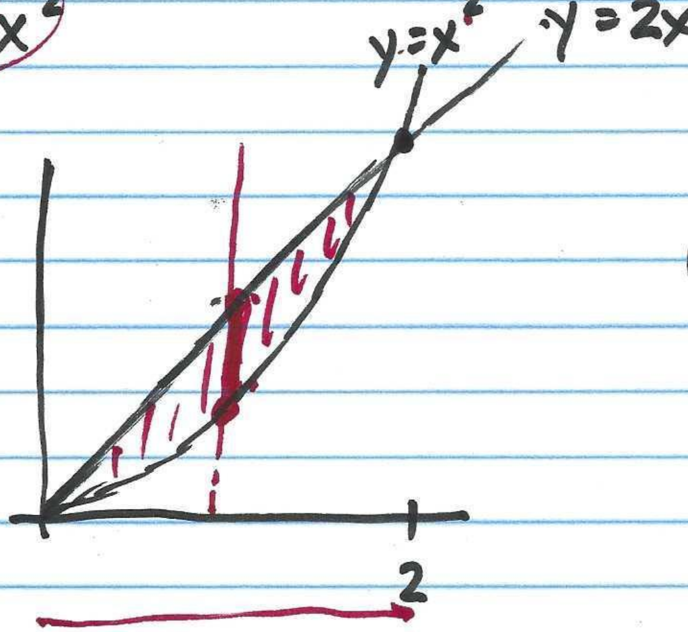
$$\int_0^1\left(\cancel{\int_0^x}\frac{\sin x}{\cancel{x}}\cdot\cancel{x}\right)dx =$$

$$\int_0^1 \sin x\,dx = \Big[-\cos x\Big]_0^1 = -\cos 1 + 1$$

$$= 1 - \cos 1$$

11

$$\int_0^2 \int_{x^2}^{2x} (4x+2)\,dy\,dx$$



$$\rightarrow \int_0^2 \Big[4xy+2y\Big]_{x^2}^{2x}\,dx$$

$$\left(8x^2+4x\right)-\left(4x^3+2x^2\right)$$

$$\underline{6x^2+4x-4x^3}$$

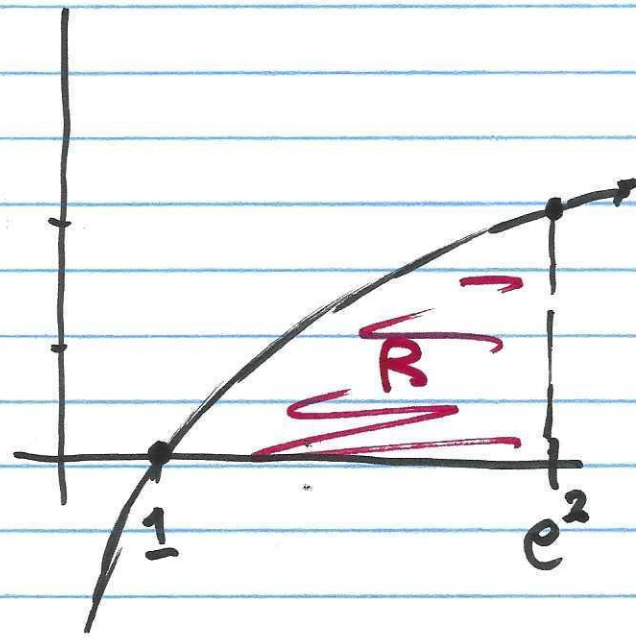
$$\int_0^2 \left(6x^2+4x-4x^3\right)dx$$

$$\Big[2x^3+2x^2-x^4\Big]_0^2 = 8$$

(12)

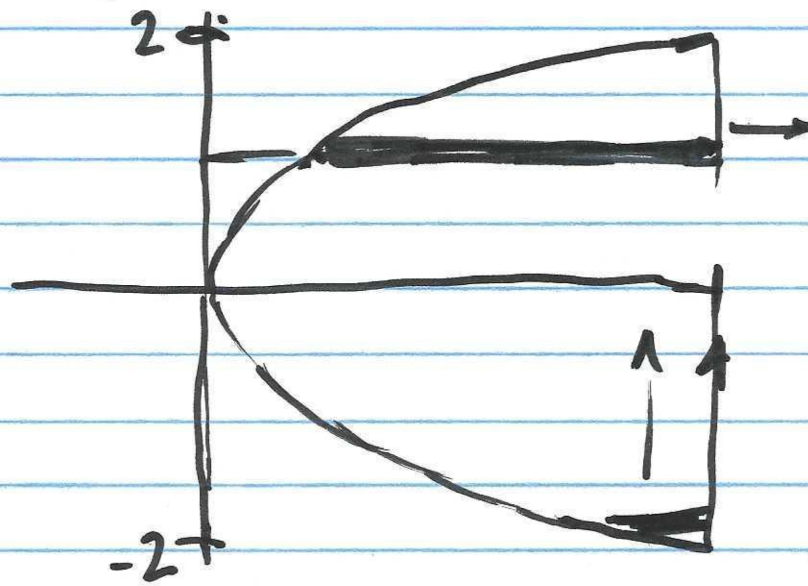
$x: \quad 1 \le x \le e^2$

$y: \quad 0 \le y \le \ln x$



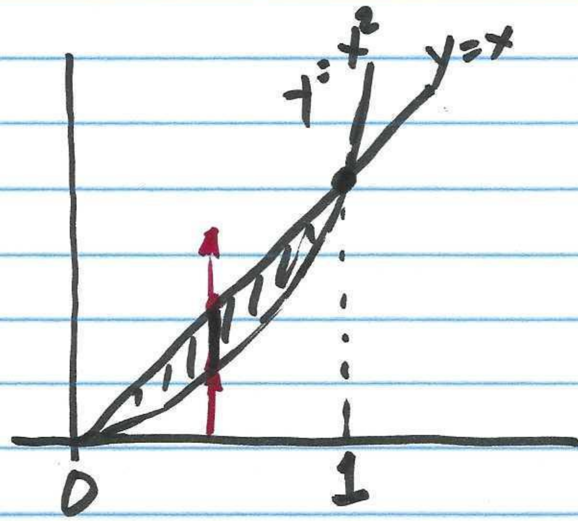
$x: \quad y^2 \le x \le 4$

$y: \quad -2 \le y \le 2$



Areas by double integration

Find area between $y=x$; $y=x^2$ in 1Q.

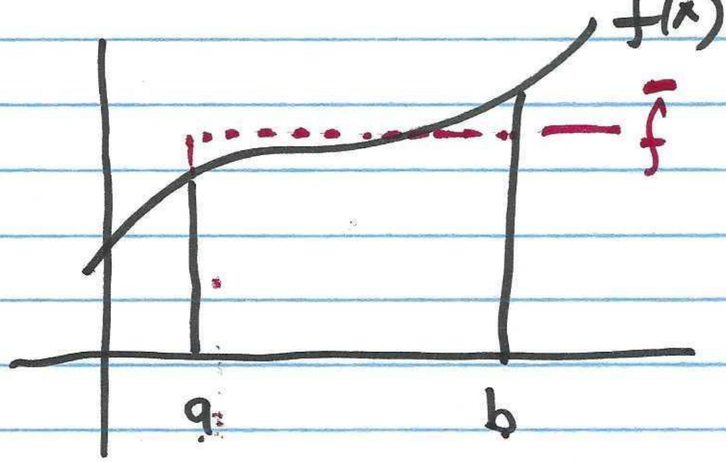


$$A=\int_0^1\int_{x^2}^{x} dy\,dx$$

$$=\int_0^1 (x-x^2)\,dx=\left[\frac{x^2}{2}-\frac{x^3}{3}\right]_0^1=\frac{1}{6}$$

Weighted average:

$y = f(x)$ over $[a,b]$

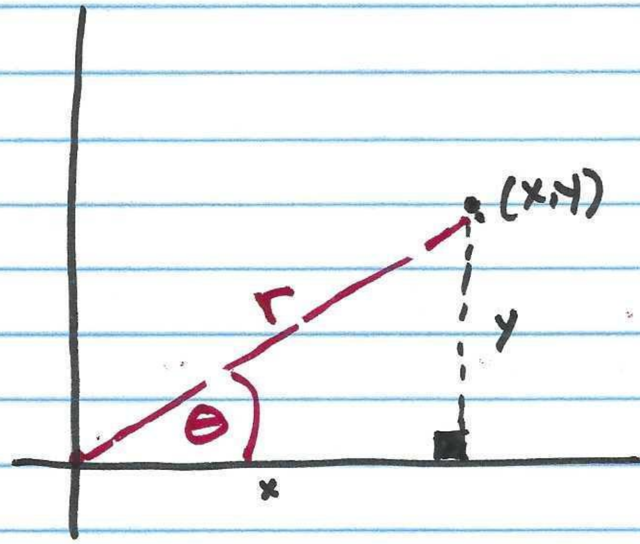


$$\bar{f}_{[a,b]} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Weighted average over region R in xy plane

$$\bar{f}_R = \frac{1}{A(R)} \cdot \iint_R f(x,y)\,dx\,dy$$

(15)



$x = r\cos\theta \qquad y = r\sin\theta$

$r = (x^2+y^2)^{1/2} \qquad \theta = \arctan\left(\frac{y}{x}\right)$