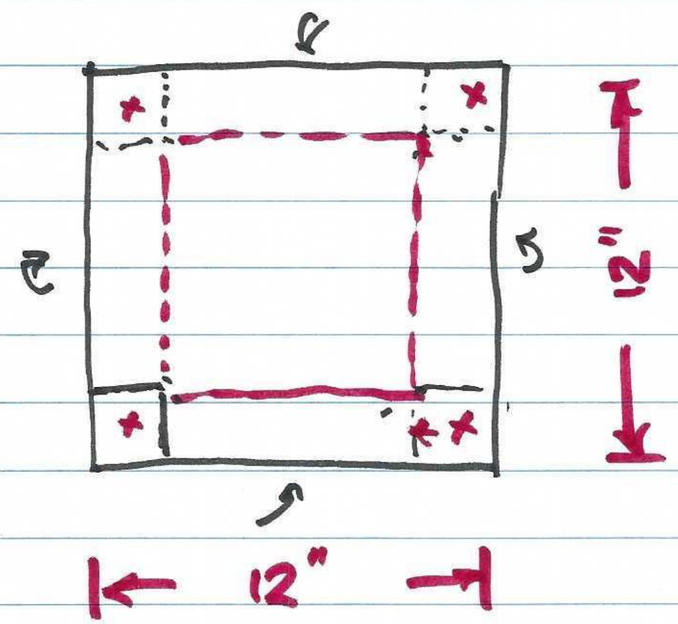
Optimization in 2 variables

1-d example

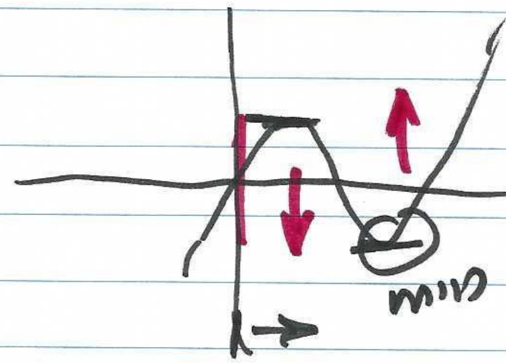
Given a 12" × 12" cardboard & you want to make a box of max volume to hold something bolts.



Objective function: Volume is function of x

Base area · Height = Vol

$(12-2x)^2 \cdot x = \text{Vol}$



$V(x) = (144 - 48x + 4x^2)x$

$= 4x^3 - 48x^2 + 144x$

$\rightarrow V'(x) = 12x^2 - 96x + 144 = 0$

$= x^2 - 8x + 12 = 0$

$= (x-6)(x-2) = 0 \Rightarrow$

crit. pts. are $x = 2, 6$

also look @ $x = 0$ - reject

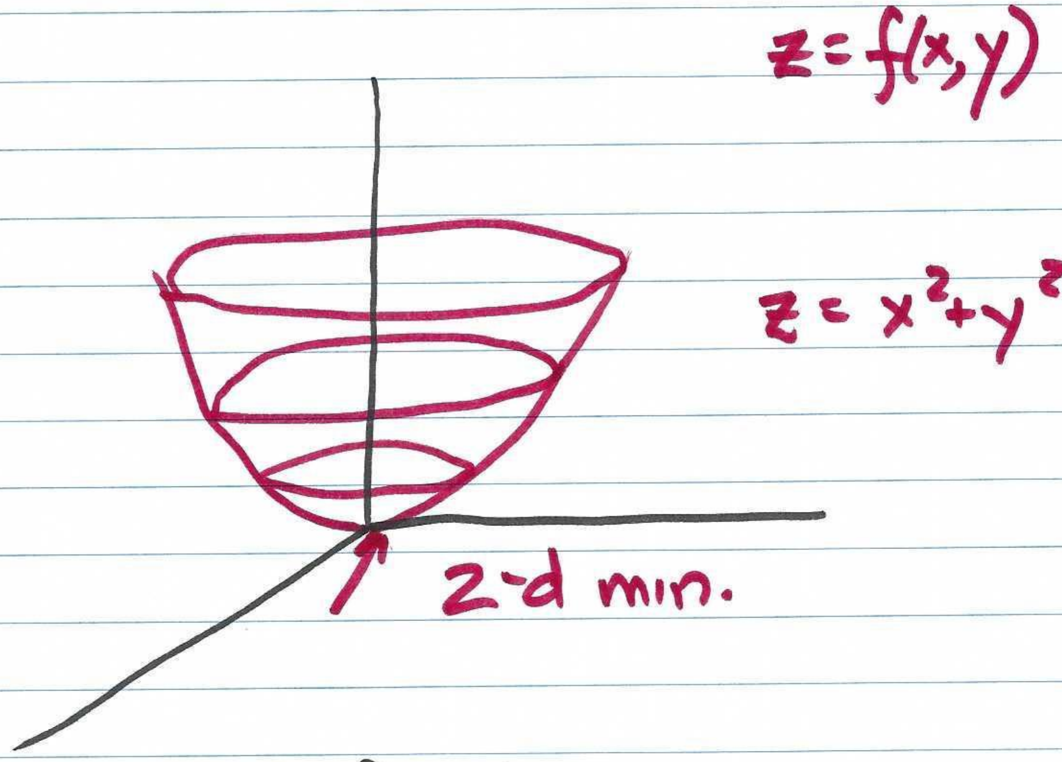
$V''(x) = 24x - 96$

$V''(2) = -48$ ✓

$V''(6) = +48$

$\Rightarrow$ pick $x = 2''$ to maximize $V(x)$

$V(2) = 4 \cdot 8 - 48 \cdot 4 + 288 = \underline{128}$? cu. in.



$z = f(x,y)$

Critical points:

(i) $f_x = 0$ & $f_y = 0$ solve together to get "candidate" points

(ii) if f_x and/or f_y DNE

Separate check: boundary of dom f

Instead of concavity test in 1-dim, we do the following:

Calculate f_{xx}, f_{yy}, & f_{xy}

If $\Delta(x,y) := f_{xx} \cdot f_{yy} - (f_{xy})^2 > 0$

proceed.

* If $\Delta(x,y) = 0$ - abandon test

* If $\Delta(x,y) < 0$ saddle point

Proceed:

Now look @ signs of f_{xx} & f_{yy}

If either is positive, candidate point is a min; if either is negative, candidate point is a max.

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Given three numbers $x, y,$ & z, if $x+y+z=12$, what is the max product xyz that can be gotten.

First, set $z = 12-x-y$

$$P(x,y,z) = xy(12-x-y)$$
$$= 12xy - x^2y - xy^2$$

① Find P_x, P_y

$$P_x = 12y - 2xy - y^2 = 0$$
$$P_y = 12x - x^2 - 2xy = 0$$

$$12y - y^2 = 2xy$$
$$12x - x^2 = 2xy$$

$$12y - y^2 = 12x - x^2$$

$$12(y-x) = y^2 - x^2 = (y-x)(y+x)$$

$x = y$

So candidate point is (x,x)

② P_{xx}, P_{yy} :

$$(12-2(x-y))^2$$

$$P_{xx} = -2y \qquad P_{xy} = 12-2x-2y$$

$$P_{yy} = -2x$$

$$\text{So } \Delta(x,y) = (-2x)(-2y) - \left[144 - 4(x-y) + (x-y)^2\right]$$

$$= 4xy - \left[\qquad \right]$$

$x = y = z = 4$ is solution.

#14 $\quad f(x,y) = x^3 + 3xy + y^3$

① Get f_x, f_y

$$f_x = 3x^2 + 3y = 0 \rightarrow 3x^2 = -3y$$

$$f_y = 3y^2 + 3x = 0 \qquad 3y^2 = -3x$$

$$x = -y^2$$

$$3x^2 = -3y$$

$$3(-y^2)^2 = -3y =$$

$3x^2 = -3y \Rightarrow \cancel{3}(-y^2)^2 = -\cancel{3}y$

$y^4 = -y \Rightarrow y^4 + y = 0 \Rightarrow y(y^3+1) = 0$

$y = 0$ or -1

Since $x = -y^2$, $x = 0$ or -1

Cand. Criti. pts are $\underline{(0,0), (-1,-1)}$

Find f_{xx}, f_{yy}, f_{xy}

$f_{xx} = 6x \quad f_{yy} = 6y \quad f_{xy} = 3$

$\Delta(x,y) = (6x)(6y) - 9 = \underline{36xy - 9}$

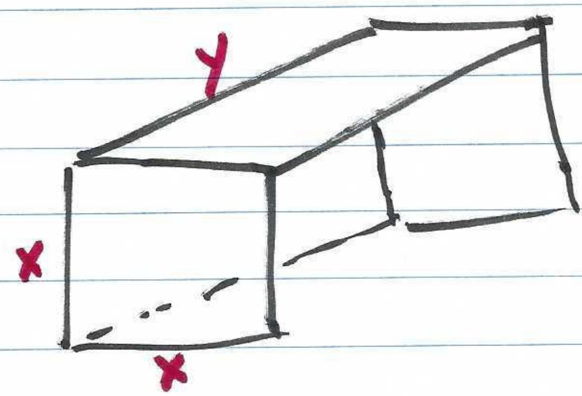
$\Delta(0,0) = -9$ saddle point

$\Delta(-1,-1) = 27$

$f_{xx}(-1,-1) = -6 \Rightarrow$ max point

$f(-1,-1) = -1 + 3 - 1 = 1$

Design beach cabana:



Back
Ends
Top

Want 250 cf

Find width/length solution that minimizes canvas area.

$V(x,y) = x^2y$

$\rightarrow A(x,y) = \underline{2x^2} + \underline{xy} + \underline{xy} = \underline{2x^2+2xy}$

$A_x = 4x+y$

$A_y = 2x$ this is really a 1-d problem

#23 $f(x,y) = y\sin x$

$f_x = y\cos x = 0 \Rightarrow y = 0$

$f_y = \sin x = 0$

$\cos x = 0 \quad x = \frac{\pi}{2} + n\pi$

$\sin x = 0 \quad x = 0 + n\pi$

no sol'n

#19 $f(x,y) = 4xy - x^4 - y^4$

$f_x = 4y - 4x^3 = 0 \qquad y - x^3 = 0$

$f_y = 4x - 4y^3 = 0 \qquad x - y^3 = 0$

$x = y^3$

$y - y^9 = 0 \Rightarrow y(1 - y^8) = 0$

$y = 0$ or 1 or -1

$x = 0 \quad 1 \quad -1$

Candidates are $(0,0), (1,1), (-1,-1)$

$f_{xx} = -12x^2$

$f_{yy} = -12y^2$

$f_{xy} = 4$

$\Delta(x,y) = 144x^2y^2 - 16$

$\Delta(0,0) = -16 \rightarrow$ saddle

$\Delta(1,1) = 128 > 0$ so ok

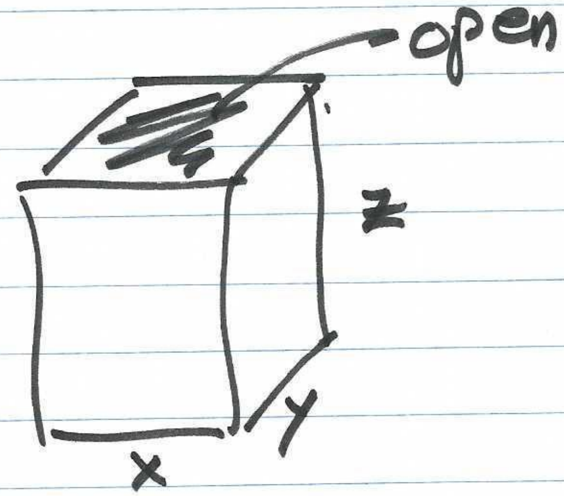
$\Delta(-1,-1) = 128 > 0$ " "

$f_{xx}(1,1) = -12 < 0$ so max

$f_{xx}(-1,-1) = -12 < 0$ so max

$(0,0)$ is saddle point

$(-1,-1)$ & $(1,1)$ are max's.



Bottom xy

Side 1 xz (2)

Side 2 yz (2)

Const $= xy + 2xz + 2yz = 12$

Vol $= xyz$

$$2(xz + yz) = 12 - xy$$

$$z = \frac{12 - xy}{2(x+y)}$$

$$\text{Vol} = xy\left(\frac{12 - xy}{2(x+y)}\right) = \frac{12xy - (xy)^2}{2(x+y)}$$

$$V(x,y) = \frac{12xy - (xy)^2}{2(x+y)}$$

$$V_x = \frac{2(x+y)(12y - 2xy^2) - (12xy - (xy)^2)(2)}{4(x+y)^2}$$