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Partial Differentiation

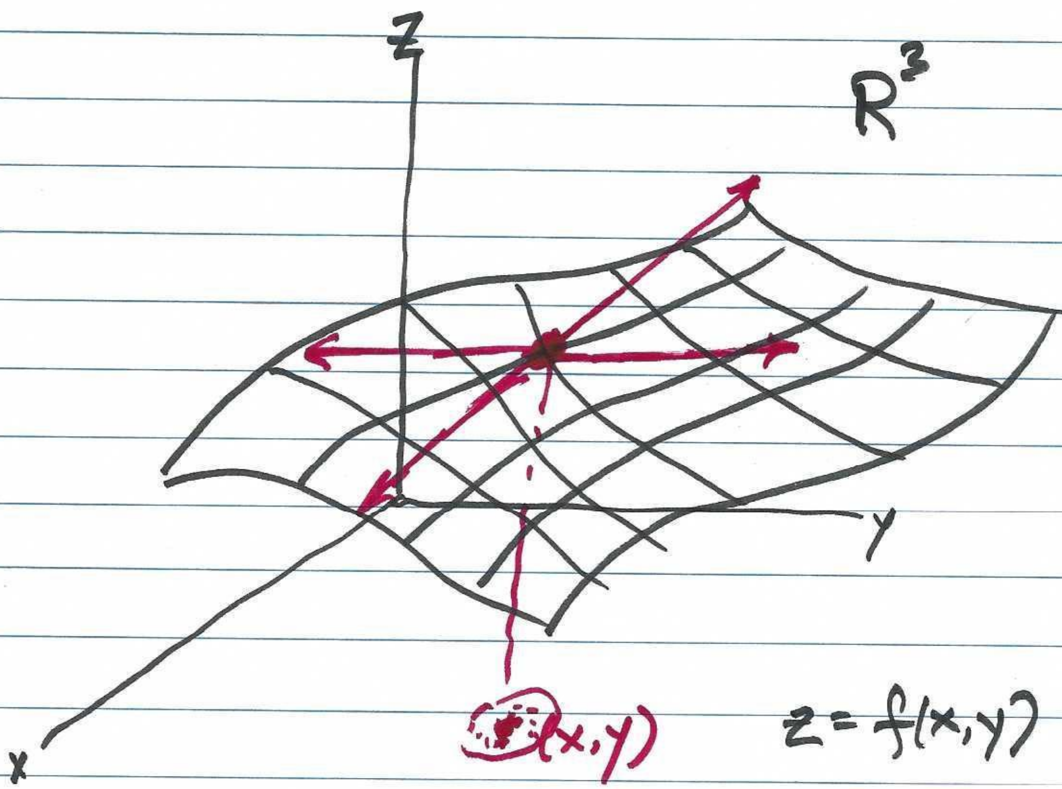
Given $f(x,y)$, we can form

$\frac{df(x)}{dx}$ $\frac{\partial f(x,y)}{\partial x}$

$$\lim_{\Delta x \to 0} \left(\frac{f(x+\Delta x, y) - f(x,y)}{\Delta x} \right) = \frac{\partial f}{\partial x} \text{ and}$$

$$\lim_{\Delta y \to 0} \left(\frac{f(x, y+\Delta y) - f(x,y)}{\Delta y} \right) = \frac{\partial f}{\partial y}$$

These are called partial derivatives.



Other notations:

$$\frac{\partial f}{\partial x} \quad f_x \quad D_x \quad \partial_x$$

Ex: $f(x,y) = x^2 + 3xy + y - 1$

Find $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ eval @ $(4,-5)$

(i) $\frac{\partial f}{\partial x} = 2x + 3y + 0 - 0$

(ii) $\frac{\partial f}{\partial y} = 0 + 3x + 1 - 0$

$$\left.\frac{\partial f}{\partial x}\right|_{(4,-5)} = 2\cdot 4 + 3\cdot(-5) = -7$$

$$\left.\frac{\partial f}{\partial y}\right|_{(4,-5)} = 3\cdot 4 + 1 = 13$$

Ex: $f(x,y) = \dfrac{2y}{y+\cos x}$

$$\frac{\partial f}{\partial x} = \left[2y\,(y+\cos x)^{-1}\right]'_x$$

$$= 2y(-1)(y+\cos x)^{-2}\cdot(-\sin x)$$

$$= \frac{2y\sin x}{(y+\cos x)^2}$$

$$\frac{\partial f}{\partial y} = 2(y+\cos x)^{-1} + (2y)(y+\cos x)^{-2}(-1)(1)$$

$$= \frac{2(y+\cos x)}{(y+\cos x)^2} - \frac{2y}{(y+\cos x)^2}$$

$$\cancel{= 2(2y)(y+\cos x)}$$

$$\frac{2(\cancel{y}+\cos x)}{(y+\cos x)^2}$$

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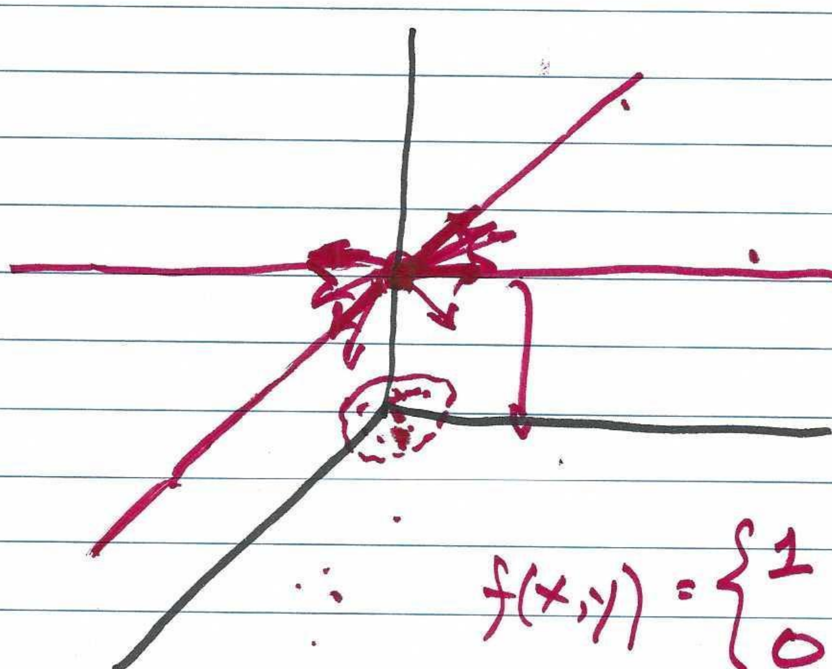
$$f(x,y,z) = x \sin(y+3z)$$

$$\frac{\partial f}{\partial x} = (1)\sin(y+3z)$$

$$\frac{\partial f}{\partial y} = x\cos(y+3z)\cdot(1) = x\cos(y+3z)$$

$$\frac{\partial f}{\partial z} = x\cos(y+3z)(3) = 3x\cos(y+3z)$$



$$f(x,y) = \begin{cases} 1 & \text{if } xy=0 \\ 0 & \text{else} \end{cases}$$

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$$\frac{\partial^2 f(x,y)}{\partial x^2} \qquad \frac{\partial^2 f(x,y)}{\partial x \partial y} \leftarrow \underline{\frac{\partial}{\partial x}\left(\frac{\partial \quad}{\partial y}\right)}$$

pure mixed

$$\boxed{\begin{array}{ll} f_{xx}(x,y) & f_{yx}(x,y) \\ \partial_{xx} & \partial_{yx} \end{array}}$$

Ex: $f(x,y) = x\cos y + ye^x$

$$\frac{\partial^2 f}{\partial x \partial y}, \quad \frac{\partial^2 f}{\partial y \partial x}, \quad \frac{\partial^2 f}{\partial x^2}, \quad \frac{\partial^2 f}{\partial y^2}$$

① $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial y}(-x\sin y + e^x) \xrightarrow{\frac{\partial}{\partial x}} \underline{(-\sin y + e^x)}$

② $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial x}(\cos y + ye^x) \xrightarrow{\partial/\partial y} (-\sin y + e^x)$

③ $\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}(\cos y + ye^x) \xrightarrow{\frac{\partial}{\partial x}} (ye^x)$

④ $\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}(-x\sin y + e^x) \xrightarrow{\frac{\partial}{\partial y}} (-x\cos y)$

$f(x,y,z) = 1 - 2xy^2z + x^2y$

$f_{yxyz} = ?$

$f_y = -4xyz + x^2$

$f_{yx} = -4yz + 2x$

$f_{yxy} = -4z$

$f_{yxyz} = -4$

$z \to z + \Delta z$

$f(x_0, y_0) \to$

Differentiable @ point in $\mathbb{R}^2$ or $\mathbb{R}^3$

Suppose $z = f(x,y)$

(*) $\Delta z = f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y + ?$

$$\boxed{\varepsilon_1 \Delta x + \varepsilon_2 \Delta y}$$

$\varepsilon_1 \to 0$, $\Delta x \to 0$, $\varepsilon_2 \to 0$, $\Delta y \to 0$

#46 $s(x,y) = \arctan\left(\frac{y}{x}\right)$

$$S_x = \left(\frac{1}{1+\frac{y^2}{x^2}}\right)\left(-\frac{y}{x^2}\right)$$

$$S_y = \left(\frac{1}{1+\frac{y^2}{x^2}}\right)\left(\frac{1}{x}\right)$$

$$S_x = \left(\frac{x^2}{x^2+y^2}\right)\cdot\left(-\frac{y}{x^2}\right) = \frac{-y}{x^2+y^2}$$

$$S_y = \left(\frac{x^2}{x^2+y^2}\right)\left(\frac{1}{x}\right) = \frac{x}{x^2+y^2}$$

$$S_{xx} = \left[-y(x^2+y^2)^{-1}\right]_x = -y\left(-1(x^2+y^2)^{-2}\right)\cdot 2x$$

$$= \frac{2xy}{(x^2+y^2)^2}$$

$$S_{yy} = \left[x(x^2+y^2)^{-1}\right]_y = x(-1)(x^2+y^2)^{-2}\cdot 2y$$

$$= \frac{-2xy}{(x^2+y^2)^2}$$ so $S_{xx} = -S_{yy}$

$$S_{xy} = \left[(-y)(x^2+y^2)^{-1}\right]_y = (-1)(x^2+y^2)^{-1} + (-y)\left(-\frac{2y}{(x^2+y^2)^2}\right)$$

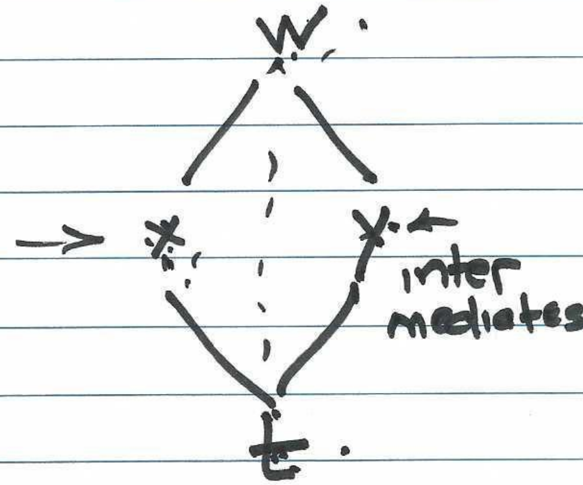
$$= -\frac{1}{x^2+y^2} + \frac{2y^2}{(x^2+y^2)^2}$$

$$= -\frac{(x^2+y^2)}{(x^2+y^2)^2} + \frac{2y^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$S_{yx} = ?$

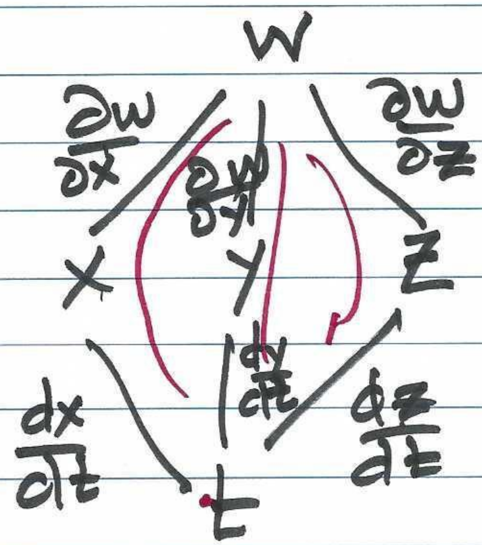
$$\left[f(g(x))\right]' = f'(g(x))\cdot g'(x)$$

$$w = f(x(t), y(t))$$



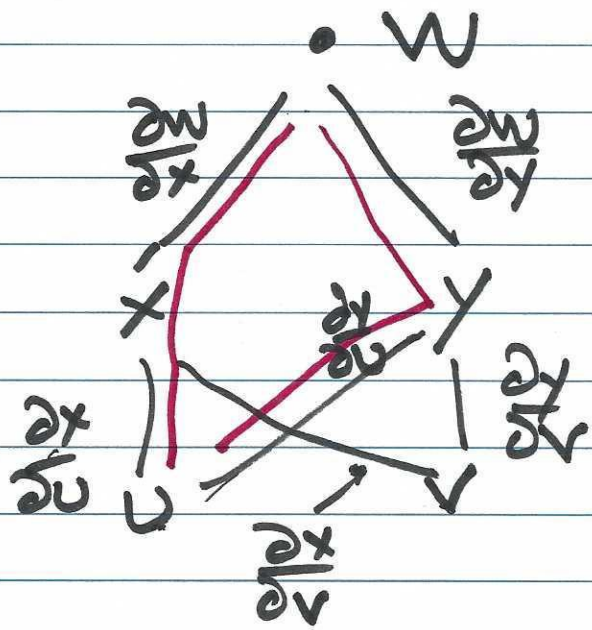
$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$$

$$w = f(x, y, z),\ x(t), y(t), z(t)$$



$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

$w = f(x,y),\ x(u,v),\ y(u,v)$



Want: $\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial u}$

#16 Want $\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}$

$r = r(x,y)$

$w = f(r,s,t),\ r = g(x,y),\ s = h(x,y),$

$t = k(x,y)$

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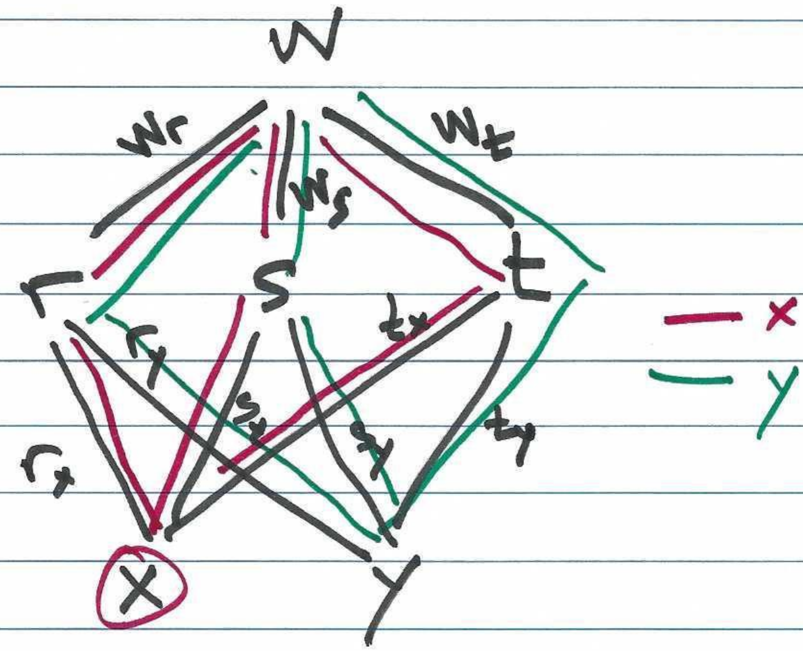
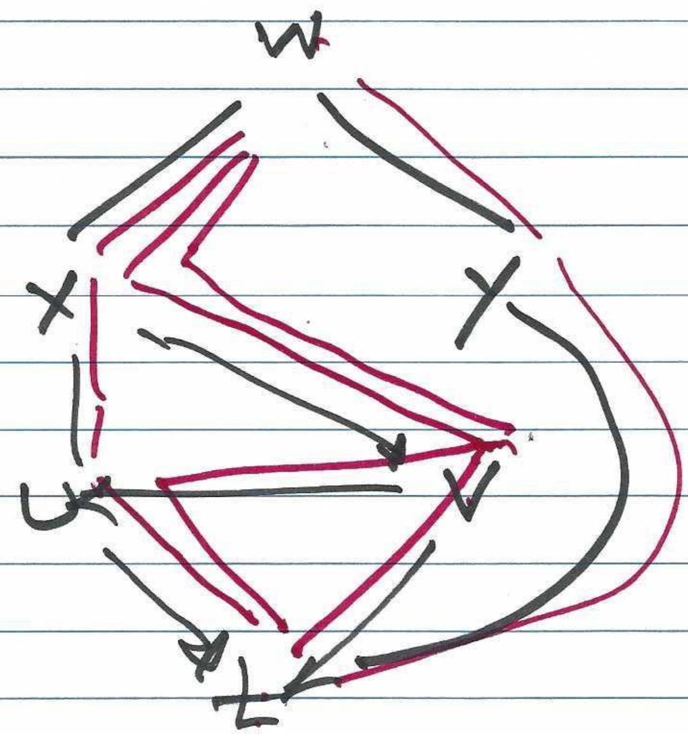


$$\frac{\partial w}{\partial x} = w_r r_x + w_s s_x + w_t t_x$$

$$\frac{\partial w}{\partial y} = w_r r_y + w_s s_y + w_t t_y$$

$w = w(x, y)$, $x(u, v)$, $v(u, t)$,

$u(t)$, $y(t)$



$$\frac{dw}{dt} = w_x x_u \frac{du}{dt} + w_x x_v v_u \frac{du}{dt} +$$

$$w_x x_v v_t + w_y \frac{dy}{dt}$$

Implicit Diff in 1-D

$$y^3 + 3yx = x^2, \quad \text{want } \frac{dy}{dx}$$

$$3y^2 \frac{dy}{dx} + 3\frac{dy}{dx}(x) + 3y = 2x$$

$$\frac{dy}{dx}\left(3y^2 + 3x\right) = 2x - 3y$$

$$\frac{dy}{dx} = \frac{2x-3y}{3y^2+3x}$$

Suppose $F(x,y)=0$ defines y implicitly as a function of x.

$$\underline{xy^2} - \sin xy = 0$$

$$0 = \frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx}$$

$$0 = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\cdot\frac{dy}{dx}$$

$$\frac{dy}{dx} = -\frac{\partial F/\partial x}{\partial F/\partial y}$$

Ex: Want $\frac{dy}{dx}$ when $y^2 - x^2 - \sin xy = 0$

$F_x: -2x - y\cos xy$

$F_y: 2y - x\cos xy$

$$\frac{dy}{dx} : ?$$

next page

$$\frac{dy}{dx} = \left(\frac{2x + y\cos xy}{2y - x\cos xy} \right)$$