4/8

Series

	Am't of "travel"
Drops 1 m	1
Rebounds .95 m	.95
Drops .95 m	.95
Rebounds (.95)(.95) m	0.9025
	↓

Question: How far has it travelled after "infinitely" many bounces

$a_0 = 1$

$a_1 = .95$

$a_2 = .95$

$a_3 = .9025$

$a_4 = .9025$

$a_5 = \vdots$

Total distance is $1 + \underline{.95} + \underline{.95} + \cdot \_\_ \_\_$

$a_0 + a_1 + a_2 + a_3 + \cdots$

$$\boxed{D = \sum_{i=0}^{\infty} a_i}$$

Want to evaluate $\rightarrow$ $D = 1 + 2(.95) + 2(.9025) + 2(\quad) \cdots$

Suppose we have a series (called a geometric series because each term is a fixed multiple of the preceding)

$$S = a_0 + \cancel{a_0 r} + \cancel{a_0 r^2} + \cancel{a_0 r^3} + \cancel{\cdots} \; \cancel{a_0 r^n} + \cancel{\cdots}$$

$\ominus$

$$rS = \cancel{r a_0} + \cancel{a_0 r^2} + \cancel{a_0 r^3} + \cancel{a_0 r^4} + \cancel{\cdots} \; \cancel{a_0 r^{n+1}} + \cancel{\cdots}$$

$S - rS = a_0$ $\quad$ $S(1-r) = a_0 \Rightarrow S = \dfrac{a_0}{1-r}$

Gen'l Formula for summing an infinite series of geometric terms

$$S = a_0 + r a_0 + r^2 a_0 + \cdots$$

$$S = \frac{a_0}{1-r}$$

1 + $2(.95) + 2(.95)^2 + 2(.95)^3 + \cdots$

$$S = .95 + .95^2 + .95^3 + \cdots$$

$a_0 = .95 \quad r = .95$

$$S = \frac{.95}{1-.95} = \frac{.95}{.05} = 19$$

$1 + 2(19)$ is total travel $= 39$ meters

(4) Σ means add the following:

Def^{n}'s Given $S = \sum_{i=0}^{\infty} a_i$ is called an infinite series. The sum is S. The general term is a_i.

There is really no operation called infinite summing.

So what we really mean by an infinite sum is the <u>limit</u> of finite sums

$$a_0 \;\big|\; a_0 + a_1 \;\big|\; a_0 + a_1 + a_2 \;\big|\; a_0 + a_1 + a_2 + a_3 \;\big|$$

$$\text{partial sum} \rightarrow \quad S_0 \qquad S_1 \qquad S_2 \qquad S_3$$

$$\boxed{\sum_{i=0}^{\infty} a_i = \lim_{n \to \infty} S_i}$$

Notation:

$$\sum_{i=0}^{\infty} a_i = \sum_{k=0}^{\infty} a_k = \sum a_k$$

New type of series: "Telescoping"

$$S = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right)$$

$$\left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) +$$

$$\cdots = 1$$

So.. $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$

Tiny change: $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

Divergence Test:

If $\lim_{n \to \infty} a_n \neq 0$ — <u>no</u> convergence

Rules for algebraic combinations:

Given $\sum a_i$, $\sum b_i$

1. $$\underbrace{\left(\sum a_i\right)}_{A} + \underbrace{\left(\sum b_i\right)}_{B} = \sum (a_i + b_i) = A + B$$

$$\sum_{n=1}^{\infty} n + \sum_{n=1}^{\infty} (-n) = 0$$

conv + conv = conv
conv + div = div
div + div = ?

2. $$\underbrace{\left(\sum a_i\right)}_{A} - \underbrace{\left(\sum b_i\right)}_{B} = \sum (a_i - b_i) = A - B$$

3. $$\underbrace{\sum a_i}_{A} \rightarrow \underbrace{\sum k a_i}_{kA}$$

$$\sum \frac{1}{n}, \quad \sum \frac{1}{n+1} \text{ but } \underline{\sum \frac{1}{n} \cdot \frac{1}{n+1}}$$

Phenomenon: You can add (or subtract) <u>any</u> <u>finite</u> number of terms of a series without affecting convergence/divergence.

Operation of Re-indexing

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1+h}^{\infty} a_{n-h} \qquad a_{(1+h)-h} = a_1$$

$$S_n = a_0 + \cancel{a_0 r} + \cancel{a_0 r^2} + \cancel{\cdots} \cancel{a_0 r^n}$$

$$r S_n = \cancel{a_0 r} + \cancel{a_0 r^2} + \cancel{a_0 r^3} + \cancel{\cdots} \; a_0 r^{n+1}$$

$$S_n - r S_n = a_0 - a_0 r^{n+1}$$

⑥

$$S_n(1-r) = a_0(1-r^{n+1})$$

$$S_n = \frac{a_0(1-r^{n+1})}{1-r}$$ <u>finite</u> geometric series

① $2 + \frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \cdots \frac{2}{3^{n-1}} + \cdots$ $\boxed{3}$

$$S_{n-1} = 2\left(\overset{a_0}{1} + \overset{a_1}{\frac{1}{3}} + \frac{1}{3^2} + \frac{1}{3^3} + \cdots \frac{1}{3^{n-1}}\right).$$

Use finite geometric series sum

$$S_{n-1} = 2\left(\frac{1\left(1-\left(\frac{1}{3}\right)^n\right)}{1-\frac{1}{3}}\right) = \cancel{\frac{\left(\frac{2}{3}\right)^n}{2/3}}$$

$$= 2\left(\frac{1-\frac{1}{3^n}}{2/3}\right) = 3\left(1-\frac{1}{3^n}\right)$$

$$S_{n-1} = 3\left(1-\frac{1}{3^n}\right)$$

$$\lim_{n\to\infty} 3\left(1-\frac{1}{3^n}\right) = \boxed{3}$$

(3) $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + - \cdots (-1)^{n-1} \cdot \frac{1}{2^{n-1}}$

(+) $\cdot\ 1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \cdots \frac{+1}{4^{n-1}}$

(−) $-\frac{1}{2} - \frac{1}{8} - \frac{1}{32} - \cdots$

1	1
2	3
3	5
4	7

$$-\left[\underline{\frac{1}{2} + \frac{1}{8} + \frac{1}{32}} + \frac{1}{128} + \cdots\right] = -\left[\underset{1}{\frac{1}{2}} + \underset{2}{\frac{1}{2^3}} + \underset{3}{\frac{1}{2^5}} + \underset{4}{\frac{1}{2^7}} + \cdots\right.$$

general term for (+) : $\frac{1}{4^{n-1}}$

$\frac{1}{2^{2n-1}}$

general term for (−) : $-\frac{1}{2^{2n-1}}$

$$\sum_{n=1}^{\infty} \frac{1}{4^{n-1}} - \sum_{n=1}^{\infty} \frac{1}{2^{2n-1}}$$

$$\frac{1}{2^{2n-1}} = \frac{1}{2}\left(\frac{1}{4^n}\right)$$

geometric w/ ratio $\frac{1}{4}$

Sum : $\frac{1}{1-1/4} = \boxed{\frac{4}{3}}$

10

$$-\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{4^n} \qquad \frac{1/4}{1-1/4} = \frac{1}{3}$$

$\downarrow$

sums to $-\frac{1}{2}\left(\frac{1}{3}\right) = -\frac{1}{6}$

Finally $1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{16} = \frac{8}{6}-\frac{1}{6} = \boxed{\frac{7}{6}}$

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7 $\sum_{n=0}^{\infty}\frac{(-1)^n}{4^n}$

$1+\left(-\frac{1}{4}\right)+\frac{1}{16}-\frac{1}{64}+\frac{1}{256}-\frac{1}{1024}+\cdots$

$(+)$ $1+\frac{1}{4^2}+\frac{1}{4^4}+\frac{1}{4^6}$

$(-)$ $-\left[\frac{1}{4}+\frac{1}{4^3}+\frac{1}{4^5}+\cdots\right]$

+ series constant ratio is $\frac{1}{16}$, initial is 1

− series constant ratio is $\frac{1}{16}$, initial is $\frac{1}{4}$

+ $\frac{1}{1-\frac{1}{16}} = \boxed{\frac{16}{15}}$

− $\frac{\frac{1}{4}}{1-\frac{1}{16}} = \frac{1}{\cancel{4}} \cdot \frac{\cancel{16}^{4}}{15} = \frac{4}{15}$

So + − − $= \frac{16}{15} - \frac{4}{15} = \frac{12}{15} = \boxed{\frac{4}{5}}$

12 $\sum_{n=0}^{\infty} \left(\frac{5}{2^n} - \frac{1}{3^n} \right) =$

$5 \sum_{n=0}^{\infty} \frac{1}{2^n} - \sum_{n=0}^{\infty} \frac{1}{3^n}$

↓ ↓

$5 \cdot \frac{1}{1-\frac{1}{2}} - \frac{1}{1-\frac{1}{3}} = 10 - \frac{3}{2} = \boxed{\frac{17}{2}}$

28 $1.\overline{414}$ = what fraction?

$$1 + 414\underbrace{\left(\frac{1}{10^3} + \frac{1}{10^6} + \frac{1}{10^9} + \cdots\right)}$$

$$1 + 414\left(\frac{\frac{1}{1000}}{\frac{999}{1000}}\right) = 1 + 414\left(\frac{1}{999}\right)$$

$$1 + \frac{414}{999} = \frac{999 + 414}{999} = \frac{1413}{999} = \frac{157}{111}$$

Fact: $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \infty$