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Sequences

term or element → $a_0, a_1, a_2, \cdots \to \infty$

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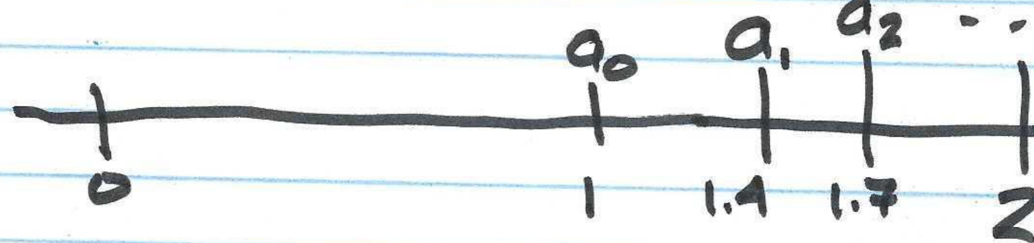
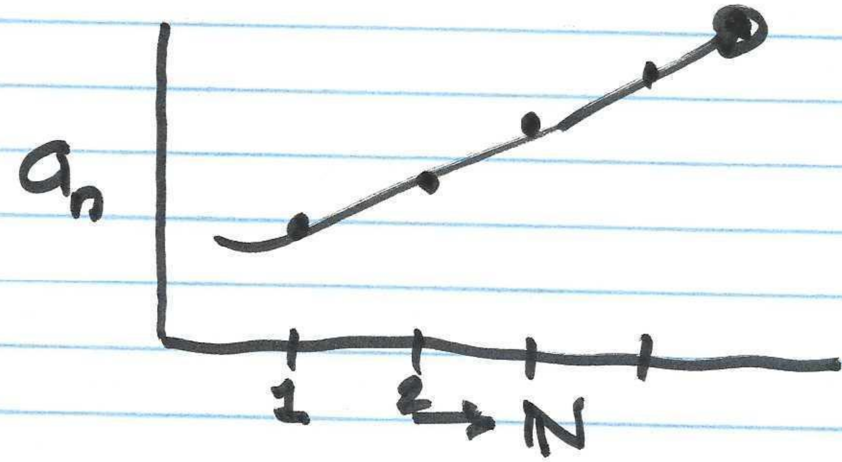
$$\{\sqrt{n}\}_{n \in \mathbb{N}} : \sqrt{1}, \sqrt{2}, \sqrt{3}, \cdots \sqrt{n} \to$$





$\{\frac{1}{n}\}_{n \in \mathbb{N}}$ $\quad a_0 = 1 \cdots a_n = \frac{1}{n} \cdots$

$a_1 = \frac{1}{2}$

$a_2 = \frac{1}{3}$

$\lim_{n\to\infty} \left(\frac{1}{n}\right) = 0$ because

Given $\varepsilon > 0$ if I can find a $N \in \mathbb{N}$ such that if $n > N$ it is true that $\left|\frac{1}{n} - 0\right| < \varepsilon$, then 0 is the limit of the sequence.

Sequences either:

(1) Converge (satisfy def^n above)

(2) Diverge $\longrightarrow$ tend to $+\infty$ or $-\infty$

$\searrow$ oscillate

$a_0 = (-1)^0 \quad a_1 = (-1)^1 \quad a_2 = (-1)^2 \ldots$

$a_n = (-1)^n$

$1, -1, 1, -1, \ldots \ldots \pm 1$

$(-2)^n = a_n$

Algebra of sequences:

1. If $\{a_n\}$ & $\{b_n\}$ are sequences, so is $\{a_n + b_n\}$. If $\{a_n\} \to A$ and $\{b_n\} \to B$ then $\{a_n + b_n\} \to A + B$

Warning: If $\{a_n + b_n\}$ converges neither $\{a_n\}$ or $\{b_n\}$ must converge.

$a_n = \{+n\}$, $b_n = \{-n\}$ then

$\{a_n + b_n\}$ converges to 0

2. Given sequences in 1 above

$$\{a_n - b_n\} \longrightarrow A - B$$

3. Constant multiple rule:

If $\{a_n\} \to A$, then $\{ka_n\} \to kA$

(4) Product Rule

With sequences as above:

$$\lim_{n\to\infty} (a_n b_n) = A \cdot B$$

(5) Quotient Rule

With usual setup

$$\lim_{n\to\infty} \left(\frac{a_n}{b_n}\right) = \frac{A}{B} \quad \underline{\text{if}} \quad \underline{\underline{B \neq 0}}$$

Ex. $a_n = \frac{1}{n^2}$ $\quad b_n = \frac{1}{n}$

A few problems:

$-1 \leq \cos n \leq +1$

(1) Find $\lim_{n\to\infty} \frac{\cos n}{n}$

$\frac{|b_n|}{n}$ $\quad 0 \leq |b_n| \leq 1 \implies 0$

2 Find $\lim \left(\frac{1}{2^n}\right)$ as $n \to \infty$

Given $\varepsilon > 0$. Want $\left|\frac{1}{2^n} - 0\right| < \varepsilon$

Need to find $N \in \mathbb{N}$ such that

$n > N, \left|\frac{1}{2^n}\right| < \varepsilon \Rightarrow \frac{1}{2^n} < \varepsilon$

$$\frac{1}{\varepsilon} < 2^n$$

$$\ln\left(\frac{1}{\varepsilon}\right) < n \ln 2$$

$\dfrac{\ln(1/\varepsilon)}{\ln 2} < n$ Choose $n > \underbrace{\dfrac{\ln(1/\varepsilon)}{\ln 2}}_{N}$

then condition is satisfied and we conclude 0 is the limit.

Th^m/ Continuous Function Th^m for Limits

Given $\{a_n\}$ if $a_n \to L$ and $f(x)$ continuous @ L and defined for all a_n, then $f(a_n) \to f(L)$.

Ex: Let $a_n = 1 - \frac{1}{n}$

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} \left(1 - \frac{1}{n}\right) = 1$$

Consider a continuous like $f(x) = x^2$

$$a_n^2 \to 1$$

$$a_n^2 = \underline{1 - \frac{2}{n} + \frac{1}{n^2}} \to 1$$

L'Hôpital's Rule ?

$$\underline{\lim_{x \to a} \frac{f(x)}{g(x)}} = \lim_{x \to a} \frac{f'(x)}{g'(x)} \to$$

$\frac{\infty}{\infty}$

What is $\lim_{n\to\infty}\left(\frac{\ln n}{n}\right) = \lim_{n\to\infty}\left(\frac{1/n}{1}\right) = 0$

Recursive Definition of a Sequence

$\underline{a_{n+2} = a_{n+1} + a_n}$ where $a_0 = 0$

$a_1 = 1$

$a_{n+2} = 1$

$a_1 + a_2 = a_3$

$1 \quad 1 \quad 2$

$a_2 + a_3 = a_4$

$1 \quad 2 = 3$

$0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots$

Fibonacci Sequence

$$a_n = \frac{\sqrt{5}}{5}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{\sqrt{5}}{5}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

(9)

(1) $a_n = \frac{1-n}{n^2}$

$a_1 = 0$

$a_2 = -1/4$

$a_3 = -2/9$

$a_4 = -3/16$

(2) $a_n = \frac{1}{n!}$

$a_1 = 1$

$a_2 = \frac{1}{2}$

$a_3 = \frac{1}{6}$

$a_4 = \frac{1}{24}$

(3) $a_1 = 1 \qquad a_{n+1} = a_n + \left(\frac{1}{2}\right)^n$

$a_2 = 1 + \left(\frac{1}{2}\right)^2 = 1\frac{1}{4}$

$a_3 = 1\frac{1}{4} + \left(\frac{1}{2}\right)^3 = 1\frac{1}{4} + \frac{1}{8} = 1\frac{3}{8}$

$a_4 = 1\frac{3}{8} + \left(\frac{1}{2}\right)^4 = 1\frac{3}{8} + \frac{1}{16} = 1\frac{7}{16}$

$\vdots$

(10)

$a_1 = 2 \quad a_2 = -1 \quad \boxed{a_{n+2} = \frac{a_{n+1}}{a_n}}$

$a_3 = \frac{a_2}{a_1} = \frac{-1}{2} = -\frac{1}{2}$

$a_4 = \frac{a_3}{a_2} = \frac{-1/2}{-1} = +\frac{1}{2}$

$a_5 = \frac{a_4}{a_3} = \frac{\frac{1}{2}}{-\frac{1}{2}} = -1$

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(21) $\underset{a_1}{1}, \underset{a_2}{5}, 9, 13, 17$ (+4, +4)

$a_n = a_1 + (n-1)4$

$= 1 + (n-1)4$

$= \underline{4n - 3}$

(23) $\underset{a_1}{\frac{5}{1}}, \frac{8}{2}, \frac{11}{6}, \frac{14}{24}, \frac{17}{120}$

$\frac{5+(n-1)3}{n!} = a_n$

$\boxed{\frac{3n+2}{n!}}$

30 $\sqrt{5/8}, \sqrt{\frac{7}{11}}, \sqrt{\frac{9}{14}}, \sqrt{\frac{11}{17}}$

$\uparrow a_1$ $\quad \uparrow a_2$

$$a_n \overset{?}{=} \sqrt{\frac{5+(n-1)2}{8+(n-1)3}}$$

$$a_n = \sqrt{\frac{2n+3}{3n+5}}$$

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31 $a_n = 2 + (0.1)^n$ Converges to 2

33 $a_n = \dfrac{1-5n^4}{n^4+8n^3}$ $\quad \dfrac{\frac{1}{n^4} - 5}{1 + \frac{8}{n}} \begin{matrix} \rightsquigarrow -5 \\ \longrightarrow 1 \end{matrix} \quad \dfrac{-5}{1} = \begin{matrix} -5 \\ 3 \end{matrix}$

37 $a_n = \dfrac{n^2 - 2n + 1}{n-1} = \dfrac{(n-1)^2}{n-1} = n-1$

diverge to $+\infty$

40 $a_n = (-1)^n \left(1 - \frac{1}{n}\right)$ diverges by oscillation

$\hookrightarrow 1$

53 $a_n = \dfrac{\ln(n+1) \to \infty}{\sqrt{n} \to \infty} = \lim_{n\to\infty} \dfrac{\frac{1}{n+1}}{\frac{1}{2\sqrt{n}}} =$

$\lim_{n\to\infty} \dfrac{2\sqrt{n}}{n+1} = \lim_{n\to\infty} \dfrac{\frac{1}{4\sqrt{n}}}{1} = \boxed{\lim_{n\to\infty} \dfrac{1}{4\sqrt{n}}} \to 0$

70 $a_n = \dfrac{n!}{2^n \cdot 3^n} = \dfrac{n!}{6^n} = ?$

>1

$\dfrac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6 \,|\, 7\cdot 8\cdot 9\cdot 10}{6\cdot 6\cdot 6\cdot 6\cdot 6\cdot 6 \,|\, 6\cdot 6\cdot 6\cdot 6} \to$

$\dfrac{120}{6^5} \cdot 1 \cdot$ (infinitely many factors > 1)

$\left(\dfrac{120}{6^5}\right)\left(\dfrac{7}{6}\right)^{n-6} < \dfrac{n!}{6^n}$

13

99 $a_n = \frac{1}{n}\int_1^n \frac{dx}{x} = \frac{1}{n}\left[\ln x\right]_1^n = \frac{\ln n}{n}$

100 $a_n = \int_1^n \frac{dx}{x^p} \quad p > 1$

$$\int x^{-p}dx = \left[\frac{x^{-p+1}}{1-p}\right]_1^n$$

$$\left[\frac{x^{1-p}}{1-p}\right]_1^n \qquad \frac{n^{1-p}}{1-p} - \frac{1}{1-p} =$$

$$\frac{n^{1-p}-1}{1-p} = \frac{1}{1-p}\left(\frac{1}{n^{p-1}} - 1\right) \to \infty$$