(10.7) Power Series

$$\frac{1}{1-x} = \underbrace{1+x+x^2+x^3+\cdots x^k+\cdots}_{\text{power series}} = \sum_{n=0}^{\infty} x^n$$

$$1 = (1-x)(1+x+x^2+x^3+\cdots)$$

$$= 1 + \underline{x - x} + \underline{x^2 - x^2} + \underline{x^3 - x^3}$$

General Power Series

$$\sum_{n=0}^{\infty} c_n x^n \quad \text{this is "centered" @ } x=0$$

$$\rightarrow \sum_{n=0}^{\infty} c_n (x-a)^n \quad \text{centered @ } x=a$$

<u>Convergence</u> $\frac{1}{1-x}$ when does it converge

$|x|<1$ this converges.

Ex: $1 - \frac{1}{2}(x-2) + \frac{1}{4}(x-2)^2 - \frac{1}{8}(x-2)^3 + \cdots$

$\sum c_n (x-2)^n$

?

$c_0 = 1$

$c_1 = -\frac{1}{2}$

$c_2 = +\frac{1}{4}$

$c_3 = -\frac{1}{8}$

$c_n = (-1)^n \left(\frac{1}{2}\right)^n$

$$\star \longrightarrow \sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{2}\right)^n (x-2)^n = \frac{2}{x}$$

$$\frac{(-1)^{\cancel{n}+1} \left(\frac{1}{2}\right)^{\cancel{n}+1} (x-2)^{\cancel{n}+1}}{(-1)^{\cancel{n}} \left(\frac{1}{2}\right)^{\cancel{n}} (x-2)^{\cancel{n}}} = \frac{(-1)(x-2)}{2}$$

This is a geometric series with initial term 1 and constant ratio $\frac{-(x-2)}{2}$

r

For convergence, $\left| \frac{-(x-2)}{2} \right| = \left| \frac{x-2}{2} \right| < 1$

$$\text{Sum} = \frac{1}{1-\frac{x-2}{2}} = \frac{1}{\frac{2-(x-2)}{2}} = \boxed{\frac{2}{x}}$$

- $x > 0$

$0! = 1$

Ex: $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = e^x$

What is the sum of the reciprocal factorials?

$$\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \cdots$$

$$1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \cdots = \sim 2.718 = \underset{?}{e}$$

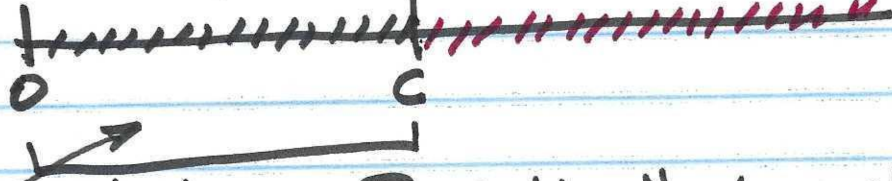
Fact: If power series $\displaystyle\sum_{n=0}^{\infty} a_n x^n$ converges when $x = c$, then it converges for all $|x| < |c|$ con

and if it diverges for $x = d$, then it diverges for all $|x| > |d|$.

Radius of convergence

x here means series converges

x here means series diverges



distance R is the "<u>radius</u> of convergence"

between 0 and R is the interval of convergence.

A power series may be differentiated term-by-term (or integrated) within its radius of convergence.

e^x converges for all x, so $R = \infty$

Consider $\dfrac{1}{1-x} = \displaystyle\sum_{n=0}^{\infty} x^n$ $\qquad \underline{-1 < x < 1} \qquad R = 1$

$|x| < 1$

Diff

$$\boxed{\frac{d}{dx}\left(\frac{1}{1-x}\right)} = \frac{d}{dx}\left(1+x+x^2+x^3+\cdots\right)$$

$$(1-x)^{-1} \rightarrow -1(1-x)^{-2} = -\frac{-1}{(1-x)^2} = \boxed{\frac{1}{(1-x)^2}}$$

RHS: $0 + 1' + 2x' + 3x^{2'} + \cdots$

$$\boxed{\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty}(n+1)x^n}$$

let $u = 1-x$ then $du = -dx$

Int

$$\int \frac{dx}{1-x} = \int \frac{-du}{u} = -\ln|u| = -\ln|1-x|$$

term-by-term integration

$$\int 1\,dx + \int x\,dx + \int x^2\,dx + \cdots$$

$\downarrow$

$$x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots$$

⑥

$-1 \le x \le 1$

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - + \cdots$$

differentiate term-by-term

$\to 1 - x^2 + x^4 - x^6 + \cdots$

constant term $= 1$
ratio $-x^2$

$$\Bigg\} \Rightarrow \frac{1}{1-(-x^2)} = \frac{1}{1+x^2}$$

What is $\int \frac{dx}{1+x^2} = \arctan x + C$

So evidently $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = \underline{\arctan x + 0}$

$$\frac{1}{1+t} = 1 - t + t^2 - t^3 + t^4 - \cdots$$

$\downarrow$

$$\frac{1}{1-(-t)}$$

$$\int_0^x \frac{dt}{1+t} = \ln(1+t)\Big|_0^x = \ln(1+x) - \underbrace{\ln 1}_{0} = \ln(1+x)$$

$$\int 1\,dt - \int t\,dt + \int t^2\,dt - \int t^3\,dt + \cdots$$

$$\left. t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \cdots \right|_0^x$$

$$-1 \le x \le 1$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + - \cdots$$

$$\boxed{\ln(1+x) \approx x} \qquad \boxed{\sin x \approx x}$$

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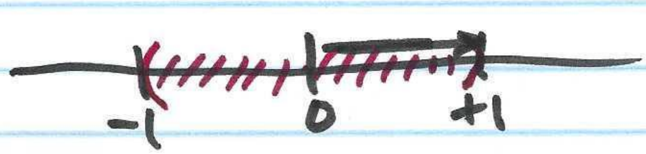
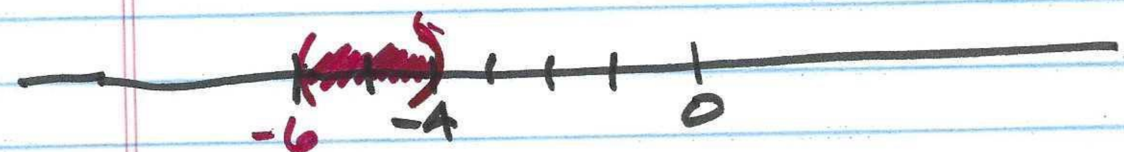
1. $$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 \cdots \quad \frac{1}{1-x}$$

$$|x| < 1$$



2 $\sum_{n=0}^{\infty}(x+5)^n$ $\qquad \frac{(x+5)^{n+1}}{(x+5)^n} = |x+5| < 1$



Interval of convergence is $(-6, -4)$

6 $\sum_{n=0}^{\infty}(2x)^n$ $\qquad \frac{(2x)^{n+1}}{(2x)^n} = \underline{|2x| < 1}$

$\underline{-\frac{1}{2} < x < \frac{1}{2}}$

⑨

T/F

① F $\int u\,dv = uv + \int v\,du$

② T $\int u\,dv + \int v\,du = uv$

③ T $\int \frac{dx}{(x-1)(x+2)} = \int \frac{dx}{x-1} - \int \frac{dx}{x+2}$!

$$\frac{1}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}$$

$$\cancel{\frac{(x+2)-(x-1)}{(x-1)(x+2)}}$$

④ F $\int \frac{dx}{(x-2)^2(x^2+1)} = \int \frac{3\,dx}{(x-2)^2} + \int \frac{5\,dx}{x^2+1}$

⑤ Write PFD $\frac{1}{(x+1)^3(x^2+2x+3)} = \_\_\_\_\_\_$

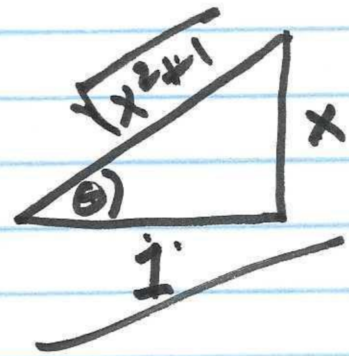
$\int e^x \sin x \, dx$

$\frac{d}{dx}$		$\int dx$
$\sin x$	$+$	e^x
$\cos x$	$-$	e^x
$-\sin x$	$+$	e^x
$-\cos x$		e^x

$$\int e^x \sin x \, dx = (\sin x)(e^x) - (\cos x)(e^x) - \int e^x \sin x \, dx$$

$$2\int e^x \sin x \, dx = e^x \sin x - e^x \cos x$$

Trig Substitution



$$\int \sqrt{x^2+1}\,dx$$

$x = \tan\theta \qquad dx = \sec^2\theta\,d\theta$

$$\int \sqrt{1+\tan^2\theta}\,\sec^2\theta\,d\theta$$

$$\int \sec^3\theta\,d\theta$$

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \operatorname{sech} x = \frac{2}{e^x+e^{-x}}$$

$$\cancel{\tanh x} = \frac{e^x+e^{-x}}{e^x-e^{-x}}$$

$$\int_0^1 \frac{dx}{\sqrt{x}} \qquad x^{-\frac{1}{2}} \xrightarrow{\int} \frac{x^{\frac{1}{2}}}{\frac{1}{2}} = 2\sqrt{x}\,\Big|_0^1$$

$$\int_0^\infty f(x)\,dx = \lim_{M\to\infty}\int_0^M f(x)\,dx$$

$$a_n = \frac{1}{n} \qquad \lim_{n\to\infty} a_n = 0$$

$$b_n = n \qquad \lim_{n\to\infty} b_n = \infty$$