Absolute Convergence

Given series $\sum a_n$, if $\sum |a_n| < \infty$ we say the series converges absolutely.

Big Th^my is: If $\sum |a_n| < \infty$, then $\sum a_n < \infty$

So.... Consider $a_n = \left(\frac{-1}{2}\right)^n$

$$1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots \quad = ?$$

We know $\sum_{n=0}^{\infty} \left|\left(\frac{-1}{2}\right)^n\right| = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{1-\frac{1}{2}} = 2$

Also... Consider $a_n = (-1)^n \cdot \frac{1}{n} \qquad n \geq 1$

$$-1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \cdots = \underline{-\ln 2}$$

But $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} = \infty$

$$\sum_{n=1}^{\infty} \frac{\sin n}{n^2} = ? \quad \underline{\text{Converges}}$$

Aside: $\sum_{n=1}^{\infty} \frac{1}{n^3}$

Look @ $\sum_{n=1}^{\infty} \left| \frac{\sin n}{n^2} \right|$ $\qquad \left| \frac{\sin n}{n^2} \right| \leq \frac{1}{n^2}$

$\sum_{n=1}^{\infty} a_n \leftarrow$ look @ a_n

Specifically look @ $\frac{a_{n+1}}{a_n}$

"Ratio Test"

Watch $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = L$

If $L = 1$ Test gives no info.

$L > 1$ Series diverges

$L < 1$ Series converges

Applying Ratio Test

Ex. $\sum_{n=0}^{\infty} \frac{2^n+5}{3^n}$

So form $\left|\frac{a_{n+1}}{a_n}\right|$ in this case $\left|\frac{\frac{2^{n+1}+5}{3^{n+1}}}{\frac{2^n+5}{3^n}}\right|$

$$= \left|\frac{\cancel{3^n}}{2^n+5} \cdot \frac{2^{n+1}+5}{\cancel{3^{n+1}}}\right| = \left|\frac{\left(2+\frac{5}{2^n}\right)}{(3)\left(1+\frac{5}{2^n}\right)}\right|$$

$$\lim_{n\to\infty}\left|\frac{2+5/2^n}{3\left(1+5/2^n\right)}\right| = \left|\frac{2}{3}\right| = \underline{2/3}$$

So we conclude $\sum \frac{2^n+5}{3^n} < \infty$

Ex. $\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2}$

Look @ $\left| \frac{a_{n+1}}{a_n} \right| = \frac{(2n+2)!}{[(n+1)!]^2} \Big/ \frac{(2n)!}{(n!)^2}$

$\boxed{\frac{(n!)^2}{(2n)!} \cdot \cancel{\frac{[(n+1)!]^2}{(2n+2)!}} \quad \text{IF} \qquad \cancel{2!\, n!\, (n+1)!\, (n+1)!}}$

$$\frac{(\cancel{2n+2})!}{(n+1)!\,(n+1)!} \cdot \frac{n!\,n!}{(\cancel{2n})!} = \frac{(2n+2)(2n+1)}{(n+1)(n+1)}$$

Note: $(2n+2)! = (2n+2)(2n+1)(2n)!$

Want $\lim_{n\to\infty} \frac{(2n+2)(2n+1)}{(n+1)^2} = \lim_{n\to\infty} \frac{(2)(2n+1)}{n+1}$

$= \lim_{n\to\infty} \frac{4n+2}{n+1} = \lim_{n\to\infty} \frac{4+\frac{2}{n}}{1+\frac{1}{n}}$

$= \underline{\underline{4}}$ so series <u>diverges</u>

<u>Root Test</u> (more sensitive than Ratio Test)

Given $\sum a_n$, calculate $\lim_{n\to\infty} \sqrt[n]{|a_n|} = r$

$\sum|a_n|$

If $r = 1$ – no conclusion

If $r > 1$ – diverges

If $r < 1$ – converges

Ex. $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$

Look @ $\lim_{n\to\infty} \sqrt[n]{\frac{n^2}{2^n}} = ?$

$$\lim_{n\to\infty} \frac{\sqrt[n]{n^2}}{\sqrt[n]{2^n}} = \lim_{n\to\infty} \frac{(\sqrt[n]{n})^2}{2}$$

Note $\lim_{n\to\infty} \sqrt[n]{n} = 1$, so

$$\lim_{n\to\infty} \frac{(\sqrt[n]{n})^2}{2} = \frac{1^2}{2} = \frac{1}{2} < 1$$ so...

converges

Ex. $\sum_{n=1}^{\infty} \frac{2^n}{n^3}$ Look @ $\lim_{n\to\infty} \sqrt[n]{\frac{2^n}{n^3}} = ?$

$$\lim_{n\to\infty} \frac{2}{\sqrt[n]{n^3}} = \lim_{n\to\infty} \frac{2}{(\sqrt[n]{n})^3} = \frac{2}{1^3}$$

$= 2$ so diverges

1. $\displaystyle\sum_{n=1}^{\infty} \underbrace{\frac{2^n}{n!}}_{a_n}$ Use ratio test

$$a_{n+1} = \frac{2^{n+1}}{(n+1)!}$$

$$\frac{a_{n+1}}{a_n} = \frac{2^{n+1}}{(n+1)!} \Big/ \frac{2^n}{n!} = \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n}$$

$$\left[(n+1)! = (n+1)n!\right]$$

$$= \frac{2^{n+1}}{(n+1)2^n} = \frac{2}{n+1}$$ so what is

$$\lim_{n\to\infty} \frac{2}{n+1} \overset{?}{=} 0$$ so <u>converges</u>

6 $\displaystyle\sum_{n=2}^{\infty} \frac{3^{n+2}}{\ln n}$ ← exponential ← logarithmic

$$\frac{\dfrac{3^{n+3}}{\ln(n+1)}}{\dfrac{3^{n+2}}{\ln n}} = \frac{3^{\cancel{n+3}}}{\ln(n+1)} \cdot \frac{\ln n}{3^{\cancel{n+2}}} = 3$$

$$\lim_{n\to\infty}\left(3\frac{\ln n}{\ln(n+1)}\right) = \lim_{n\to\infty}(3)\left(\frac{1/n}{1/n+1}\right)$$

$$3\lim_{n\to\infty}\left(\frac{n+1}{n}\right) = 3\lim_{n\to\infty}\left(1+\frac{1}{n}\right) = 3\cdot 1 = 3$$

10 $\sum_{n=1}^{\infty} \frac{4^n}{(3n)^n}$ Look@ $\lim_{n\to\infty} \sqrt[n]{\frac{4^n}{(3n)^n}}$

$$\sqrt[n]{\frac{4^n}{(3n)^n}} = \frac{\sqrt[n]{4^n}}{\sqrt[n]{(3n)^n}} = \frac{4}{3n}$$

$$\lim_{n\to\infty} \frac{4}{3n} \to 0$$

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70 $\sum_{n=1}^{\infty} \frac{2^{n^2}}{n!} \sim \frac{2^{n \cdot n}}{n!} \overset{?}{=} \frac{(2^n)^n}{n!}$

Use Ratio Test: $\frac{(2^{n+1})^{n+1}}{(n+1)!} \Big/ \frac{(2^n)^n}{n!}$

$$\frac{(2^{n+1})^{n+1}}{(n+1)\cancel{!}} \cdot \frac{\cancel{n!}}{(2^n)^n} = \left(\frac{1}{n+1}\right)\left(\frac{(2^{n+1})^n \cdot 2^{n+1}}{(2^n)^n}\right)$$

$$= \left(\frac{1}{n+1}\right)\left(2^n \cdot 2^{n+1}\right) = \left(\frac{1}{n+1}\right)\left(4^n \cdot 2\right)$$

$$= \frac{2 \cdot 4^n}{n+1} \rightarrow \infty$$

10.6 Alternating Series ; Conditional Convergence

$$\sum_{n=0}^{\infty} (-1)^n a_n \qquad a_n \geq 0 \quad \lim_{n\to\infty} |a_n| = 0$$

Series that converges, but not absolutely, is called a conditionally convergent series.

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - + \cdots = \ln 2$$

$$|1| + \left|-\frac{1}{2}\right| + \left|\frac{1}{3}\right| + \left|-\frac{1}{4}\right| + \cdots$$

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \rightarrow \infty$$

An absolutely convergent series may be re-arranged without affecting convergence. A conditionally convergent series may not.

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + - \cdots$$

$$\overbrace{\underbrace{1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \cdots}_{\text{diverges}}}^{10} \quad - \frac{1}{2} + \cdots -$$