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Integral Test

Given a series $\sum a_n$, the integral

$$\int_1^{\infty} f(n)\,dn$$ converges or diverges along

with the series. Note $a_n = f(n)$

Classic example is $\sum_{n=1}^{\infty} \frac{1}{n}$ – so look @

$\int_1^{\infty} \frac{1}{x}\,dx$ (because $\frac{1}{x}$ eval @ n is $\frac{1}{n} = a_n$).

Then since $\int_1^{\infty} \frac{dx}{x} = \ln x \Big|_1^{\infty} = \infty$

we say $\sum \frac{1}{n}$ diverges.

3b $\sum_{n=1}^{\infty} n\tan\left(\frac{1}{n}\right)$

$\infty \cdot 0$

First Pass: $\lim_{n\to\infty} n\tan\left(\frac{1}{n}\right) \Rightarrow$

$$\lim_{n\to\infty} \frac{\tan\left(\frac{1}{n}\right)}{\frac{1}{n}} = \left[\frac{0}{0}\right] \quad \frac{\sec^2\left(\frac{1}{n}\right)\cdot\cancel{\left(-\frac{1}{n^2}\right)}}{\cancel{\left(-\frac{1}{n^2}\right)}}$$

What happens to $\sec^2\left(\frac{1}{n}\right)$ as $n \to \infty$?

" " " $\sec^2(x)$ as $x \to 0$? $= 1$

$$\frac{1}{\cos^2 x}$$

We see $a_n \not\to 0$ as $n \to \infty$. But it must for series to converge. So no!

14. $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ $\quad f(x) = \frac{x}{x^2+1}$

Look @ $\int_1^{\infty} \frac{x\,dx}{x^2+1} =$ $\quad$ Let $u = x^2+1$, $du = 2x\,dx$

$$\frac{1}{2}\int_2^{\infty} \frac{du}{u} = \frac{1}{2}\Big[\ln u\Big]_2^{\infty} = \infty$$

So no convergence.

Comparison Tests

Given $\sum a_n$, $\sum b_n$, if $0 \leq a_n \leq b_n$

1) $\sum a_n$ diverges what about $\sum b_n$? Diverges

2) $\sum b_n$ converges what about $\sum a_n$? Converges

Application of Comparison Test:

1. $$\sum_{n=1}^{\infty} \frac{5}{5n-1} > \sum_{n=1\,?}^{\infty} \frac{1}{n}$$

$\uparrow$ a_n $\qquad$ $\uparrow$ b_n

$$\frac{5}{5n-1} = \frac{1}{n-\frac{1}{5}} > \frac{1}{n}$$

But $\sum \frac{1}{n}$ diverges, so then $\sum \frac{5}{5n-1}$ does, too

2. $$\sum_{n=1}^{\infty} \frac{1}{n!} = 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \cdots$$

$$\sum_{n=0}^{\infty} \frac{1}{2^n} \qquad 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} \cdots = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}$$

$$a_n = \frac{1}{n!} \qquad b_n = \frac{1}{2^n}$$

$a_n \le b_n$ $\qquad$ $\frac{1}{1 \cdot 2 \cdot 3 \cdot 4 \cdots n} = a_n$ conv

$\frac{1}{2 \cdot 2 \cdot 2 \cdots 2} = b_n$ conv

After $n=3$ $a_n \le b_n$

$\boxed{0! = 1}$

$$\sum_{n=0}^{\infty} \frac{1}{n!} = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \dots = e^1$$

Limit Comparison Test. $\sum a_n, \sum b_n$

1. If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$ and $c > 0$ then

   $\sum a_n$; $\sum b_n$ conv/diverge together

2. If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ Conclude: if $\sum b_n < \infty$

   then $\sum a_n < \infty$.

3. If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum b_n = \infty$, then

   $\sum a_n = \infty$

Ex.: $\frac{3}{4} + \frac{5}{9} + \frac{7}{16} + \frac{9}{25} = \sum_{n=1}^{\infty} \frac{2n+1}{(n+1)^2} =$

$$\sum_{n=1}^{\infty} \overset{a_n}{\frac{2n+1}{n^2+2n+1}} = \sum a_n$$

Good comparison test series is $\sum_{n=1}^{\infty} \overset{b_n}{\frac{1}{n}} = \sum b_n$

$$\frac{a_n}{b_n} = \frac{2n+1}{n^2+2n+1} \Big/ \frac{1}{n} = \frac{2n^2+n}{n^2+2n+1}$$

Want $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2n^2+n}{n^2+2n+1} = ?$

$$\lim_{n\to\infty} \frac{2+\frac{1}{n}}{1+\frac{2}{n}+\frac{1}{n^2}} \longrightarrow \frac{2}{1} = 2.$$

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Ex: $\displaystyle\sum_{n=1}^{\infty} \overbrace{\frac{1}{2^n-1}}^{a_n}$ Pick $\displaystyle\sum_{n=1}^{\infty} \overbrace{\frac{1}{2^n}}^{b_n}$ ← converges

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\frac{1}{2^n-1}}{\frac{1}{2^n}} = \lim_{n\to\infty} \frac{2^n}{2^n-1}$$

$$\lim_{n\to\infty} \frac{2^n}{2^n-1} \stackrel{?}{=} \frac{1}{1-\frac{1}{2^n}} \rightsquigarrow 1$$

Conclude $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n-1} < \infty$, since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n}$ does

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① $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+30}$ Know $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$ $\frac{1}{n^p}$

$$\frac{1}{n^2+30} \leq \frac{1}{n^2}$$

Converges by direct comparison.

5 $$\sum_{n=1}^{\infty} \frac{\cos^2 n}{n^{3/2}} \text{ converges because } \sum_{n=1}^{\infty} \frac{1}{n^{3/2}} < \infty$$

$$\frac{\cos^2 n}{n^{3/2}} \leq \frac{1}{n^{3/2}}$$ p-series with $p = 3/2$

9 $$\sum_{n=1}^{\infty} \overbrace{\frac{n-2}{n^3-n^2+3}}^{a_n} \qquad \sum_{n=1}^{\infty} \overbrace{\frac{1}{n^2}}^{b_n} < \infty$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n-2}{n^3-n^2+3} \cdot n^2 =$$

$$\lim_{n\to\infty} \frac{n^3-2n^2}{n^3-n^2+3} = \lim_{n\to\infty} \frac{1-\frac{2}{n}}{1-\frac{1}{n}+\frac{3}{n^3}}$$

$= 1$ So orig. series $\sum a_n < \infty$.

⑨

⑳ $\displaystyle\sum_{n=1}^{\infty} \overset{a_n}{\frac{1+\cos n}{n^2}}$ compare w/ $\displaystyle\sum_{n=1}^{\infty} \overset{b_n}{\frac{2}{n^2}}$

$$\underset{\text{conv}}{\frac{1+\cos n}{n^2}} \leq \underset{\text{conv}}{\frac{2}{n^2}}$$

㉛ $\displaystyle\sum_{n=1}^{\infty} \overset{a_n}{\frac{1}{1+\ln n}}$ compare to $\overset{b_n}{\frac{1}{n}}$

$\frac{\infty}{\infty}$

$$\lim_{n\to\infty} \frac{1}{1+\ln n} \Big/ \frac{1}{n} = \lim_{n\to\infty} \frac{n}{1+\ln n} = $$

$$\lim_{n\to\infty} \frac{1}{1/n} = n = \infty$$

Conclude from Limit Comparison (case 3)

that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1+\ln n} = \infty$

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(40) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n+3^n}{3^n+4^n}$

n	1	2	3
	$\frac{5}{7}$	$\frac{13}{25}$	$\frac{35}{91}$

$$\frac{2^n+3^n}{3^n+4^n} \le \frac{3^n+3^n}{3^n+4^n} \le \frac{3^n+3^n}{3^n+3^n} = 1$$

$$\frac{\left(\frac{2}{3}\right)^n+1}{1+\left(\frac{4}{3}\right)^n} = \frac{\left(\frac{2}{3}\right)^n}{1+\left(\frac{4}{3}\right)^n} + \frac{1}{1+\left(\frac{4}{3}\right)^n}$$

$\downarrow$

$\frac{1}{2^n}$ $\quad$ $\frac{1}{1+2^n}$ $\quad$ HOLD

$\cancel{\frac{2^n+3^n}{3^n+4^n}}$ Since $\sum \frac{1}{2^n} < \infty$

and $\frac{1}{2^n} \ge \frac{1}{1+2^n}$,

we know $\sum \frac{1}{1+2^n}$ converges

$$\sum \frac{1}{1+(4/3)^n}$$

$$\frac{1}{(4/3)^n} > \frac{1}{1+(\frac{4}{3})^n}$$

$$\uparrow$$

$$\sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n = \frac{3/4}{1-3/4} = 3$$

Ultimately $\sum_{n=1}^{\infty} \frac{2^n+3^n}{3^n+4^n} < \infty$

Sum $\sum_{n=1}^{\infty} \frac{1}{4n^2-1} = \frac{1}{2}$

$$\frac{1}{4n^2-1} = \frac{1}{2}\left[\frac{1}{2n-1} - \frac{1}{2n+1}\right]$$

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$$\frac{1}{2}\sum_{n=1}^{\infty}\left[\frac{1}{2n-1}-\frac{1}{2n+1}\right]=\frac{1}{2}$$

$$\underbrace{\left(1-\frac{1}{3}\right)}_{n=1}+\underbrace{\left(\frac{1}{3}-\frac{1}{5}\right)}_{n=2}+\underbrace{\left(\frac{1}{5}-\frac{1}{7}\right)}_{n=3}+\cdots$$